RMO 2025b

Non-KV and Non-JNV

Exercise 1 (RMO 2025b P1,).

(a) Let $n \geq 3$ be an integer. Find a configuration of n lines in the plane which has exactly

(i) $n-1$ distinct points of intersection;

(ii) n distinct points of intersection.

(b) Give configurations of n lines that have exactly $n+1$ distinct points of intersection for (i) $n = 8$ and (ii) $n = 9$.

Walkthrough —

(a) For part (a)(i), consider $n-1$ parallel lines and another line perpendicular to all of them.

(b) For part (a)(ii), consider $n-1$ lines concurrent at a point and another line not passing through that point.

(c) For part (b)(ii), consider three lines forming a triangle, together with the three lines forming its medians, and three lines forming the medial triangle.

(d) For part (b)(i), consider the above configuration and remove one median.

Solution 1. Consider the configuration of $n-1$ parallel lines and another line $\mathscr{L}$ which is perpendicular to some of these parallel lines. Note that $\mathscr{L}$ is also perpendicular to all the other parallel lines. It follows that this is a configuration as required in (a)(i) since all the parallel lines intersect the line perpendicular to them at distinct points, giving exactly $n-1$ points of intersection.

For (a)(ii), consider $n-1$ lines concurrent at a point O. Next, consider a point P other than O. We will specify a line passing through P and show that this line together with the $n-1$ lines concurrent at O gives a configuration as required in (a)(ii). Note that there is only one line passing through O and P. Also note that there are precisely $n-1$ lines passing through P, such that each of these lines is parallel to one of the $n-1$ lines concurrent at O.

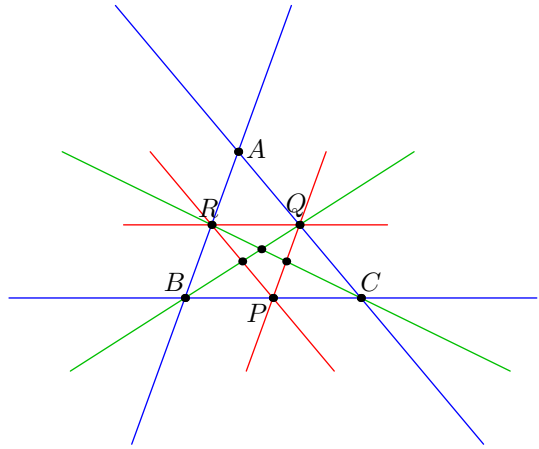


Figure 1: RMO 2025a P1, **??**(b)(i): Configuration with 8 lines and 9 intersection points.

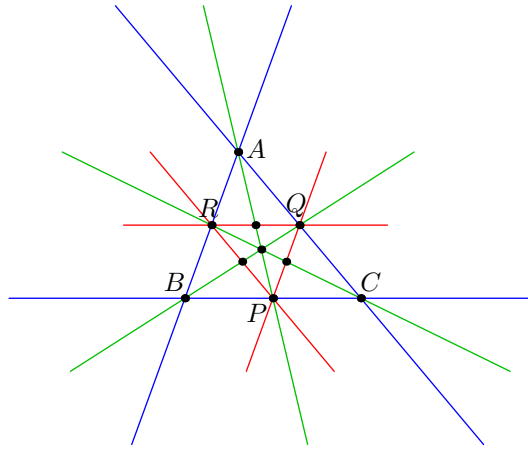


Figure 2: RMO 2025a P1, **??**(b)(ii): Configuration with 9 lines and 10 intersection points.

Let $\mathscr{L}$ denote a line passing through P which is not parallel to any of the $n-1$ lines concurrent at O, and which does not pass through O. Note that $\mathscr{L}$ intersects each of the $n-1$ lines concurrent at O at a point distinct from O, and $\mathscr{L}$ intersects the $n-1$ distinct lines concurrent at O at $n-1$ distinct points. Hence, the configuration of $n-1$ lines concurrent at O together with the line $\mathscr{L}$ gives a configuration as required in (a)(ii) with exactly n points of intersection.

For (b)(i), consider the configuration shown in Fig. 1. Here, we have 8 lines and exactly 9 points of intersection.

For (b)(ii), consider the configuration shown in Fig. 2. Here, we have 9 lines and exactly 10 points of intersection.

More precisely, we begin with 3 lines such that no two are parallel and no three are concurrent, and their 3 points of intersection are the vertices of a triangle ABC as in Fig. 2. Let P denote the midpoint of side BC, Q the midpoint of side CA, and R the midpoint of side AB. Next, we draw the line passing through P, Q, the line passing through Q, R, and the line passing through R, P. This gives a total of 6 lines with exactly 6 points of intersection. Indeed, the 3 original lines intersect at 3 points, and PQ, QR, RP are parallel to AB, BC, CA respectively, so there are precisely 6 points of intersection, which are the vertices of triangles ABC and PQR.

Next, we draw the line passing through A, P, the line passing through B, Q, and the line passing through C, R. This gives a total of 9 lines. Note that PQ is parallel to AB, and R is the midpoint of AB, showing that $PQAR$ and $PQRB$ are parallelograms. Hence, AP, RQ bisect each other, and BQ, RP also bisect each other. Similarly, CR, PQ bisect each other. Observing that AP, BQ, CR are the medians of $\triangle ABC$, it follows that they concur at the centroid of $\triangle ABC$. This gives a total of 9 lines with exactly 10 points of intersection, as required for (b)(ii).

To obtain a configuration for (b)(i), we remove one of the lines AP, BQ, CR from the above configuration. ■

Exercise 2 (RMO 2025b P2,). Let a, b, c be distinct nonzero real numbers satisfying

$$a + \frac{2}{b} = b + \frac{2}{c} = c + \frac{2}{a}.$$

Determine the value of $|a^2b + b^2c + c^2a|$.

Walkthrough —

(a) Divide the given relation by $\sqrt{2}$ and obtain suitable distinct nonzero real numbers x, y, z satisfying

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}.$$

(b) Let the common value of the above three expressions be λ.

(c) Consider the function $f\colon \mathbb{R}\setminus\{\lambda\} \to \mathbb{R}$, defined by

$$f(\xi) = \frac{1}{\lambda - \xi} \quad \text{for all } \xi \in \mathbb{R}\setminus\{\lambda\}.$$

(d) Show that for any $\xi \in \{x, y, z\}$, the elements $\xi, f(\xi), f(f(\xi))$ lie in $\mathbb{R}\setminus\{\lambda\}$, and that the relation $f(f(f(\xi))) = \xi$ holds.

(e) Prove that this yields a quadratic polynomial $p(\xi)$ in ξ such that $(\lambda^2 - 1)p(\xi)$ vanishes at x, y, z.

(f) Deduce that $\lambda = 1$ or $\lambda = -1$.

(g) Show that it suffices to consider the case $\lambda = 1$.

(h) In the case $\lambda = 1$, express y, z in terms of x, and compute the value of $|x^2y + y^2z + z^2x|$.

Solution 2. Let x, y, z be real numbers defined by

$$x = \frac{a}{\sqrt{2}}, y = \frac{b}{\sqrt{2}}, z = \frac{c}{\sqrt{2}}.$$

Note that x, y, z are also distinct nonzero real numbers. Using the given relation, we have

$$x + \frac{1}{y} = y + \frac{1}{z} = z + \frac{1}{x}.$$

Let this common value be λ, that is, λ denotes the real number which is equal to any of

$$x + \frac{1}{y}, y + \frac{1}{z}, z + \frac{1}{x}.$$

Consider the function $f\colon \mathbb{R}\setminus\{\lambda\} \to \mathbb{R}$, defined by

$$f(\xi) = \frac{1}{\lambda - \xi} \quad \text{for all } \xi \in \mathbb{R}\setminus\{\lambda\}.$$

Since y, z, x are nonzero real numbers, it follows that none of x, y, z is equal to λ, that is, x, y, z are elements of $\mathbb{R}\setminus\{\lambda\}$. Using the definition of f, we obtain

$$f(x) = y, f(y) = z, f(z) = x.$$

Note that $x, f(x), f(f(x)), y, f(y), f(f(y)), z, f(z), f(f(z))$ are elements of $\mathbb{R}\setminus\{\lambda\}$, and that

$$f(f(f(x))) = x, f(f(f(y))) = y, f(f(f(z))) = z$$

hold. If ξ is an element of $\mathbb{R}$ such that $\xi, f(\xi), f(f(\xi))$ lie in $\mathbb{R}\setminus\{\lambda\}$ and $f(f(f(\xi))) = \xi$ holds, then we have

$$f(\xi) = \frac{1}{\lambda - \xi},$$

The content posted here and at this blog by Evan Chen are quite useful.

$$\begin{aligned} f(f(\xi)) &= f\left(\frac{1}{\lambda-\xi}\right) \\ &= \frac{1}{\lambda-\frac{1}{\lambda-\xi}} \\ &= \frac{\lambda-\xi}{\lambda(\lambda-\xi)-1}, \\ f(f(f(\xi))) &= f\left(\frac{\lambda-\xi}{\lambda(\lambda-\xi)-1}\right) \\ &= \frac{1}{\lambda-\frac{\lambda-\xi}{\lambda(\lambda-\xi)-1}} \\ &= \frac{\lambda(\lambda-\xi)-1}{(\lambda^2-1)(\lambda-\xi)-\lambda}, \end{aligned}$$

and hence, we obtain

$$\frac{\lambda(\lambda-\xi)-1}{(\lambda^2-1)(\lambda-\xi)-\lambda} = \xi,$$

which yields

$$\lambda^2 - 1 - \lambda\xi = \xi(\lambda^2-1)(\lambda-\xi) - \lambda\xi,$$

which is equivalent to

$$(\lambda^2-1)(\xi^2 - \xi\lambda + 1) = 0.$$

Note that the polynomial $(\lambda^2-1)(\xi^2-\xi\lambda+1)$ in the variable ξ has degree 2 unless $\lambda^2-1=0$. Since x, y, z are distinct roots of this polynomial, it follows that $\lambda^2-1=0$, that is, $\lambda = 1$ or $\lambda = -1$.

Note that it suffices to consider the case $\lambda = 1$, since the case $\lambda = -1$ can be reduced to this case by replacing a, b, c by $-a, -b, -c$ respectively (and replacing x, y, z by $-x, -y, -z$ respectively, as a consequence), and replacing λ by $-\lambda$. Also note that replacing a, b, c by $-a, -b, -c$ respectively does not change the value of $|a^2b + b^2c + c^2a|$. Thus, we consider the case $\lambda = 1$. In this case, we have

$$x + \frac{1}{y} = 1, y + \frac{1}{z} = 1, z + \frac{1}{x} = 1.$$

From these, we obtain

$$y = f(x) = \frac{1}{1-x}, \quad z = f(f(x)) = \frac{1-x}{1(1-x)-1} = 1 - \frac{1}{x}.$$

Using these, we have

$$\begin{aligned} x^2y + y^2z + z^2x &= x^2 \cdot \frac{1}{1-x} + \left(\frac{1}{1-x}\right)^2 \cdot \left(1-\frac{1}{x}\right) + \left(1-\frac{1}{x}\right)^2 \cdot x \\ &= \frac{x^2}{1-x} + \frac{1}{(1-x)^2} \cdot \frac{x-1}{x} + \frac{(1-x)^2}{x} \end{aligned}$$

$$
\begin{aligned}
&= \frac{x^2}{1-x} - \frac{1}{x(1-x)} + \frac{(1-x)^2}{x} \\
&= \frac{x^3 - 1 + (1-x)^3}{x(1-x)} \\
&= \frac{-3x^2 + 3x}{x(1-x)} \\
&= \frac{3x(1-x)}{x(1-x)} \\
&= 3,
\end{aligned}
$$

where the final equality follows using that $x \neq 0, x \neq \lambda, \lambda = 1$. Therefore, we have

$$|a^2b + b^2c + c^2a| = |2^{3/2}(x^2y + y^2z + z^2x)| = |2^{3/2} \cdot 3| = 3 \cdot 2^{3/2} = 6\sqrt{2}.$$

■

Solution 3. Using the given condition, we obtain

$$
\begin{aligned}
a - b &= \frac{2}{bc}(b - c), \\
b - c &= \frac{2}{ca}(c - a), \\
c - a &= \frac{2}{ab}(a - b),
\end{aligned}
$$

which implies that

$$(a-b)(b-c)(c-a) = \frac{8}{(abc)^2}(a-b)(b-c)(c-a).$$

Since a, b, c are distinct, it follows that

$$|abc| = 2\sqrt{2}.$$

Also, from the given condition, we have

$$\frac{ab+2}{b} = \frac{bc+2}{c} = \frac{ca+2}{a}.$$

Let this common value be μ. Note that

$$
\begin{aligned}
a(ab+2) + b(bc+2) + c(ca+2) &= \mu(ab + bc + ca), \\
c(ab+2) + a(bc+2) + b(ca+2) &= \mu(ab + bc + ca)
\end{aligned}
$$

hold, which yields

$$a^2b + b^2c + c^2a = 3abc.$$

Therefore, we have

$$|a^2b + b^2c + c^2a| = 3|abc| = 3 \cdot 2\sqrt{2}.$$

This completes the solution. ■

Exercise 3 (RMO 2025b P4,). Prove that there do not exist positive rational number x and y such that

$$x + y + \frac{1}{x} + \frac{1}{y} = 2025.$$

Walkthrough —

(a) Consider the problem of showing that the given equation has no solutions in positive integers.

(b) Determine and compare the denominators of both sides of

$$x + \frac{1}{x} = 2025 - y - \frac{1}{y}.$$

(c) Does the above provide any insight into the original problem?

Solution 4. On the contrary, suppose there exist positive rational numbers x and y such that

$$x + y + \frac{1}{x} + \frac{1}{y} = 2025.$$

Let a, b, c, d be positive integers such that $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where $\gcd(a, b) = 1$ and $\gcd(c, d) = 1$. Note that

$$x + \frac{1}{x} = \frac{a^2 + b^2}{ab}, \quad \text{and} \quad y + \frac{1}{y} = \frac{c^2 + d^2}{cd}.$$

Observe that

$$\frac{a^2 + b^2}{ab} = 2025 - \frac{c^2 + d^2}{cd} = \frac{2025cd - (c^2 + d^2)}{cd},$$

with $\gcd(a^2 + b^2, ab) = 1$ and

$$\gcd(2025cd - (c^2 + d^2), cd) = \gcd(c^2 + d^2, cd) = 1.$$

This implies that

$$ab = cd.$$

It follows that

$$a^2 + b^2 + c^2 + d^2 = 2025cd.$$

Using the integers a, b have no common prime factors, it follows that at most one of a and b is divisible by 3. Similarly, at most one of c and d is divisible by 3. Therefore, among the four integers a, b, c, d, at most two of them are divisible by 3. If none of them is divisible by 3, then

$$a^2 + b^2 + c^2 + d^2 \equiv 4 \pmod 3 \equiv 1 \pmod 3,$$

which is impossible since 3 divides 2025. This shows that at least one of a, b, c, d is divisible by 3. Using $ab = cd$, it follows that at least two of a, b, c, d are divisible by 3. Using the above, we obtain that exactly two of a, b, c, d are divisible by 3. This implies that

$$a^2 + b^2 + c^2 + d^2 \equiv 2 \pmod 3,$$

which is again impossible since 3 divides 2025.

This proves that there do not exist positive rational numbers x and y such that

$$x + y + \frac{1}{x} + \frac{1}{y} = 2025.$$

■

Exercise 4 (RMO 2025b P6,). Let $p(x)$ be a nonconstant polynomial with integer coefficients, and let $n \geq 2$ be an integer such that no term of the sequence

$$p(0), p(p(0)), p(p(p(0))), \ldots$$

is divisible by n. Show that there exist integers a, b such that $0 \leq a < b \leq n-1$ and n divides $p(b) - p(a)$.

Walkthrough —

(a) Apply the pigeonhole principle to show that some two terms of the given sequence are congruent modulo n.

(b) Let i and j be positive integers such that $i < j$ and the i-th and j-th terms of the sequence are congruent modulo n.

(c) Reduce the arguments of these two terms modulo n to obtain integers a and b such that $0 \leq a, b \leq n-1$ and n divides $p(b) - p(a)$.

(d) Show that $a \neq b$. It could be useful to assume that the integer i as above is the smallest with this property.

Solution 5. Define the sequence

$$p^{(0)}(0), p^{(1)}(0), p^{(2)}(0), p^{(3)}(0), \ldots$$

by

$$p^{(n)}(0) \coloneqq \begin{cases} 0 & \text{if } n = 0, \\ p(p^{(n-1)}(0)) & \text{if } n \geq 1. \end{cases}$$

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By the pigeonhole principle, some two among the following $n+1$ integers

$$p^{(1)}(0), p^{(2)}(0), \ldots, p^{(n+1)}(0)$$

are congruent modulo n. Let i denote the smallest positive integer such that there exists a positive integer $j > i$ with

$$p^{(i)}(0) \equiv p^{(j)}(0) \pmod{n}.$$

Let $0 \leq a \leq n-1$ and $0 \leq b \leq n-1$ be integers such that

$$a \equiv p^{(i-1)}(0) \pmod{n}, \quad b \equiv p^{(j-1)}(0) \pmod{n}.$$

Note that n divides $p(b) - p(a)$. We claim that $a \neq b$. Suppose, for the sake of contradiction, that $a = b$. It follows that

$$p^{(i-1)}(0) \equiv p^{(j-1)}(0) \pmod{n}.$$

Using $j - 1 > i - 1$ and the minimality of i, we obtain $i = 1$. This implies that

$$p^{(0)}(0) \equiv p^{(j-1)}(0) \pmod{n}.$$

Since $p^{(0)}(0) = 0$ and $j - 1 > 0$, we have that n divides $p^{(j-1)}(0)$, contradicting the given condition. This implies that $a \neq b$.

Reordering if necessary, we may assume that $a < b$. Consequently, there exist integers $0 \leq a < b \leq n-1$ such that n divides $p(b) - p(a)$. ■