RMO 2025a

KV or JNV

Exercise 1 (RMO 2025a P1,). Solve the following system of equations in nonnegative integers $a_1, a_2, \ldots, a_8$ where $a_i \neq 1$ for $i = 1, \ldots, 8$:

$$\begin{aligned} a_1a_2 &= a_3 + a_4, \\ a_3a_4 &= a_5 + a_6, \\ a_5a_6 &= a_7 + a_8, \\ a_7a_8 &= a_1 + a_2. \end{aligned}$$

Walkthrough —

(a) Observe that $(a-1)(b-1) = ab - a - b + 1$ holds for any integers a and b.

(b) Using this identity, rewrite the system of equations conveniently.

Solution 1. Let $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ be a solution of the system of equations. Note that

$$\begin{aligned} (a_1-1)(a_2-1) &= a_1a_2 - a_1 - a_2 + 1, \\ (a_3-1)(a_4-1) &= a_3a_4 - a_3 - a_4 + 1, \\ (a_5-1)(a_6-1) &= a_5a_6 - a_5 - a_6 + 1, \\ (a_7-1)(a_8-1) &= a_7a_8 - a_7 - a_8 + 1. \end{aligned}$$

Since $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ satisfies the system of equations, we obtain

$$(a_1-1)(a_2-1) + (a_3-1)(a_4-1) + (a_5-1)(a_6-1) + (a_7-1)(a_8-1) = 4.$$

Since $a_i \neq 1$ for any $i = 1, \ldots, 8$, we have $a_i - 1 \neq 0$ for any $i = 1, \ldots, 8$. Using that $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ satisfies the system of equations, it follows that one of $a_1, \ldots, a_8$ is equal to 0 if and only if all of $a_1, \ldots, a_8$ are equal to 0. Hence, if some a_i is not equal to 0, then all of $a_1, \ldots, a_8$ are greater than or equal to 2, and in this case, we have

$$(a_1-1)(a_2-1) \geq 1,$$

$$(a_3-1)(a_4-1)\geq 1,$$
$$(a_5-1)(a_6-1)\geq 1,$$
$$(a_7-1)(a_8-1)\geq 1,$$

which implies that all of $a_1,\ldots,a_8$ are equal to 2. Also note that $(0,0,0,0,0,0,0,0)$, $(2,2,2,2,2,2,2,2)$ are solutions of the system of equations. This shows that the solutions of the system of equations in nonnegative integers where $a_i\neq 1$ for $i=1,\ldots,8$ are

$$(0,0,0,0,0,0,0,0),\quad (2,2,2,2,2,2,2,2).$$

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Exercise 2 (RMO 2025a P2,). Let a,b,c be positive real numbers satisfying $abc=1$. Prove that

$$\frac{2a^2}{a^3+1}+\frac{2b^2}{b^3+1}+\frac{2c^2}{c^3+1}\leq a^2+b^2+c^2.$$

Walkthrough —

(a) Apply the AM-GM inequality to reduce to proving that $\sqrt{a}+\sqrt{b}+\sqrt{c}\leq a^2+b^2+c^2$.

(b) Substituting $x=\sqrt{a}$, $y=\sqrt{b}$, $z=\sqrt{c}$, it suffices to prove that $x+y+z\leq x^4+y^4+z^4$ for positive real numbers x,y,z satisfying $xyz=1$.

(c) Conclude by applying the rearrangement inequality.

Solution 2. Applying the AM-GM inequality, we have

$$\begin{aligned}\frac{2a^2}{a^3+1}+\frac{2b^2}{b^3+1}+\frac{2c^2}{c^3+1}&\leq \frac{2a^2}{2\sqrt{a^{3/2}}}+\frac{2b^2}{2\sqrt{b^{3/2}}}+\frac{2c^2}{2\sqrt{c^{3/2}}}\\&=\sqrt{a}+\sqrt{b}+\sqrt{c}.\end{aligned}$$

Therefore, it suffices to prove that

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\leq a^2+b^2+c^2.$$

Writing $x=\sqrt{a}$, $y=\sqrt{b}$, $z=\sqrt{c}$, we have $xyz=1$ and we need to prove that

$$x+y+z\leq x^4+y^4+z^4,$$

which is equivalent to

$$xyz(x+y+z)\leq x^4+y^4+z^4.$$

The content posted here and at this blog by Evan Chen are quite useful.

Applying the rearrangement inequality two times, we obtain

$$\begin{aligned} x^4 + y^4 + z^4 &\geq x^2y^2 + y^2z^2 + z^2x^2 \\ &\geq xy \cdot yz + yz \cdot zx + zx \cdot xy \\ &= xyz(x + y + z). \end{aligned}$$

This completes the proof. ■

Exercise 3 (RMO 2025a P4,). A frog is initially at $(0, 0)$ and it reaches $(n, 2)$, $n \geq 1$, using the following moves in any order several times:

1. $R = (1, 0)$, that is, if the frog is at (a, b) it goes to $(a + 1, b)$;
2. $U = (0, 1)$, that is, if the frog is at (a, b) it goes to $(a, b + 1)$;
3. $D = (1, 1)$, that is, if the frog is at (a, b) it goes to $(a + 1, b + 1)$.

In how many ways can the frog go from $(0, 0)$ to $(n, 2)$, $n \geq 1$, using the above steps subject to the condition that steps of the type UU, DD are forbidden? (For example, for $n = 3$, $RDUR, DRD$ are admissible paths, while $DDR, RUURR$ are not.)

> **Walkthrough** —
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> **(a)**

Solution 3.

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Exercise 4 (RMO 2025a P6,). Define the sequence $\langle a_0, a_1, a_2, \dots \rangle$ as follows: $a_0 = 49$ and $a_n = 10^{2^n} a_{n-1} - 1$ for $n \geq 1$. Show that $s(a_n^2) = n^2 + n + 7$ for all $n \geq 0$, where $s(m)$ denotes the sum of digits in base 10 representation of a nonnegative integer m.

> **Walkthrough** —
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> **(a)**

Solution 4. For a nonnegative integer n, let $P(n)$ denote the statement that

$$2a_n \leq 10^{2^{n+1}},$$

the integer a_n^2 is congruent to 1 modulo 10, and

$$s(a_n^2) = n^2 + n + 7, \quad s(10^{2^{n+1}} - 2a_n) = 2n + 2.$$

Using $a_0 = 49$ and $a_0^2 = 2401$, it follows that $P(0)$ is true. Assume that $P(n)$ is true for some nonnegative integer n. We will show that $P(n+1)$ is true. By the definition of the sequence, we have

$$a_{n+1} = 10^{2^{n+1}} a_n - 1.$$

This gives

$$\begin{aligned} 10^{2^{n+2}} - 2a_{n+1} &= 10^{2^{n+2}} - 2(10^{2^{n+1}} a_n - 1) \\ &= 10^{2^{n+2}} - 2 \cdot 10^{2^{n+1}} a_n + 2 \\ &= 10^{2^{n+1}} (10^{2^{n+1}} - 2a_n) + 2. \end{aligned}$$

Using the induction hypothesis, we have

$$2a_{n+1} \leq 10^{2^{n+2}}.$$

Moreover, we have

$$\begin{aligned} s(10^{2^{n+2}} - 2a_{n+1}) &= s(10^{2^{n+1}} (10^{2^{n+1}} - 2a_n) + 2) \\ &= s(10^{2^{n+1}} - 2a_n) + s(2) \\ &= 2n + 2 + 2 \\ &= 2(n+1) + 2. \end{aligned}$$

Since $a_{n+1} \equiv -1 \pmod{10}$, we have $a_{n+1}^2 \equiv 1 \pmod{10}$. Also note that

$$\begin{aligned} a_{n+1}^2 &= (10^{2^{n+1}} a_n - 1)^2 \\ &= 10^{2^{n+2}} a_n^2 - 2 \cdot 10^{2^{n+1}} a_n + 1 \\ &= 10^{2^{n+2}} (a_n^2 - 1) + 10^{2^{n+1}} (10^{2^{n+1}} - 2a_n) + 1. \end{aligned}$$

Using $0 \leq 10^{2^{n+1}} - 2a_n \leq 10^{2^{n+1}}$, we obtain

$$s(a_{n+1}^2) = s(a_n^2 - 1) + s(10^{2^{n+1}} - 2a_n) + 1$$

Using the induction hypothesis, we have

$$\begin{aligned} s(a_{n+1}^2) &= (n^2 + n + 7 - 1) + (2n + 2) + 1 \\ &= (n+1)^2 + (n+1) + 7. \end{aligned}$$

This shows that $P(n+1)$ is true. By the principle of mathematical induction, we conclude that $P(n)$ is true for all nonnegative integers n. ■