# RMO 2024b

## Non-KV and Non-JNV

**Exercise 1** (RMO 2024b P1, ). Let $n > 1$ be a positive integer. Call a rearrangement $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ *nice* if for every $k = 2, 3, \ldots, n$, we have that $a_1 + a_2 + \cdots + a_k$ is not divisible by $k$.

1. If $n > 1$ is odd, prove that there is no nice arrangement of $1, 2, \ldots, n$.
2. If $n$ is even, find a nice arrangement of $1, 2, \ldots, n$.

**Solution 1.** Note that if $n > 1$ is an odd positive integer, then for any rearrangement $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$, the integer

$$a_1 + \cdots + a_n = 1 + 2 + \cdots + n = n \times \frac{n+1}{2}$$

is divisible by $n$. This proves the first part.

For any even positive integer $n$, a nice rearrangement of $1, 2, \ldots, n$ is provided by the Claim below.

> **Claim —** For any even positive integer $n$,
>
> $$2, 1, 4, 3, 6, 5, 8, 7, \ldots, n, n-1$$
>
> is a nice rearrangement of $1, 2, \ldots, n$.

*Proof of the Claim.* Note that the Claim holds for $n = 2$. Let $n$ be an even positive integer satisfying the Claim. Consider the rearrangement

$$2, 1, 4, 3, 6, 5, 8, 7, \ldots, n, n-1, n+2, n+1$$

of $1, 2, \ldots, n+2$. Note that the sum of its first $n+1$ terms is equal to

$$(n+1) \times \frac{n+2}{2} + 1,$$

which is not a multiple of $n+1$. Also note that the sum of the integers $1, 2, \ldots, n+2$ is equal to

$$(n+2) \times \frac{n+3}{2},$$

which is not a multiple of $n+2$. By the induction hypothesis, it follows that the rearrangement

$$2, 1, 4, 3, 6, 5, 8, 7, \ldots, n, n-1, n+2, n+1$$

of $1, 2, \ldots, n+2$ is nice. This proves the Claim. □

■

**Exercise 2** (RMO 2024b P2, ). For a positive integer $n$, let $R(n)$ be the sum of the remainders when $n$ is divided by $1, 2, \ldots, n$. For example, $R(4) = 0+0+1+0 = 1$, $R(7) = 0+1+1+3+2+1+0 = 8$. Find all positive integers such that $R(n) = n-1$.

**Walkthrough** —

**(a)** Note that

$$\begin{aligned}
R(1) &= 0,\\
R(2) &= 0,\\
R(3) &= 0+1+0\\
&= 1,\\
R(4) &= 0+0+1+0\\
&= 1,\\
R(5) &= 0+1+2+1+0\\
&= 4,\\
R(6) &= 0+0+0+2+1+0\\
&= 3,\\
R(7) &= 0+1+1+3+2+1+0\\
&= 8,\\
R(8) &= 0+0+2+0+3+2+1+0\\
&= 8,\\
R(9) &= 0+1+0+1+4+3+2+1+0\\
&= 12,\\
R(10) &= 0+0+1+2+0+4+3+2+1+0\\
&= 13
\end{aligned}$$

holds.

**(b)** Does the above help?

**Solution 2.** For positive integers $n$ and $k$, denote by $r(n,k)$ the remainder obtained upon dividing $n$ by $k$. Note that any integer $k \geq 4$,

$$R(2k) \geq r(2k, k-1) + r(2k, k+1) + r(2k, k+2) + \cdots + r(2k, 2k-1)$$



The content posted here and at this blog by Evan Chen are quite useful.

$$\begin{aligned}
&= 2 + (k-1) + (k-2) + \cdots + 1 \\
&= 2 + \frac{k(k-1)}{2} \\
&\geq 2k
\end{aligned}$$

holds. Also note that any integer $k \geq 3$,

$$\begin{aligned}
R(2k+1) &\geq r(2k+1,k) + r(2k+1,k+1) \\
&\quad + r(2k+1,k+2) + \cdots + r(2k+1,2k-1) + r(2k+1,2k) \\
&= 1 + k + (k-1) + \cdots + 2 + 1 \\
&= 1 + \frac{k(k+1)}{2} \\
&\geq 2k+1
\end{aligned}$$

holds. Note that

$$\begin{aligned}
R(1) &= 0, \\
R(2) &= 0, \\
R(3) &= 0 + 1 + 0 \\
&= 1, \\
R(4) &= 0 + 0 + 1 + 0 \\
&= 1, \\
R(5) &= 0 + 1 + 2 + 1 + 0 \\
&= 4, \\
R(6) &= 0 + 0 + 0 + 2 + 1 + 0 \\
&= 3
\end{aligned}$$

holds. This proves that the positive integers $n$ satisfying $R(n) = n - 1$ are precisely $1, 5$. ■

**Exercise 3** (RMO 2024b P4, ). Let $a_1, a_2, a_3, a_4$ be real numbers such that $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$. Show that there exist $i, j$ with $1 \leq i < j \leq 4$, such that $(a_i - a_j)^2 \leq \frac{1}{5}$.

**Solution 3.** Reordering $a_1, a_2, a_3, a_4$ if necessary, we assume that $a_1 \leq a_2 \leq a_3 \leq a_4$ holds. Put

$$\begin{aligned}
a_2 &= a_1 + x, \\
a_3 &= a_2 + y, \\
a_4 &= a_3 + z.
\end{aligned}$$

Note that

$$a_2^2 = a_1^2 + x^2 + 2a_1x,$$

$$\begin{aligned}
a_3^2 &= (a_1 + x + y)^2 \\
&= a_1^2 + x^2 + y^2 + 2a_1x + 2a_1y + 2xy, \\
a_4^2 &= (a_1 + x + y + z)^2 \\
&= a_1^2 + x^2 + y^2 + z^2 + 2a_1x + 2a_1y + 2a_1z + 2xy + 2yz + 2zx
\end{aligned}$$

hold. This gives

$$\begin{aligned}
&a_1^2 + a_2^2 + a_3^2 + a_4^2 \\
&= a_1^2 \\
&\quad + a_1^2 + x^2 + 2a_1x \\
&\quad + a_1^2 + x^2 + y^2 + 2a_1x + 2a_1y + 2xy \\
&\quad + a_1^2 + x^2 + y^2 + z^2 + 2a_1x + 2a_1y + 2a_1z + 2xy + 2yz + 2zx \\
&= 4a_1^2 + 3x^2 + 2y^2 + z^2 + 2a_1(3x + 2y + z) + 4xy + 2yz + 2zx \\
&= \left(2a_1 + \frac{3x + 2y + z}{2}\right)^2 - \frac{(3x + 2y + z)^2}{4} \\
&\quad + 3x^2 + 2y^2 + z^2 + 4xy + 2yz + 2zx \\
&= \left(2a_1 + \frac{3x + 2y + z}{2}\right)^2 - \left(\frac{9}{4}x^2 + y^2 + \frac{1}{4}z^2 + 3xy + yz + \frac{3}{2}zx\right) \\
&\quad + 3x^2 + 2y^2 + z^2 + 4xy + 2yz + 2zx \\
&= \left(2a_1 + \frac{3x + 2y + z}{2}\right)^2 + \frac{3}{4}x^2 + y^2 + \frac{3}{4}z^2 + xy + yz + \frac{1}{2}zx.
\end{aligned}$$

If $x > \frac{1}{\sqrt{5}}, y > \frac{1}{\sqrt{5}}, z > \frac{1}{\sqrt{5}}$ holds, then we would obtain

$$\begin{aligned}
1 &> \left(2a_1 + \frac{3x + 2y + z}{2}\right)^2 + \left(\frac{3}{4} + 1 + \frac{3}{4} + 1 + 1 + \frac{1}{2}\right)\frac{1}{5} \\
&= \left(2a_1 + \frac{3x + 2y + z}{2}\right)^2 + 1 \\
&\geq 1,
\end{aligned}$$

which yields $1 > 1$, which is impossible. This shows that at least one of $x, y, z$ is less than or equal to $\frac{1}{\sqrt{5}}$. ■

**Exercise 4** (RMO 2024b P6, ). Let $n \geq 2$ be a positive integer. Call a sequence $a_1, a_2, \ldots, a_k$ of integers an *$n$-chain* if $1 = a_1 < a_2 < \cdots < a_k = n$, $a_i$ divides $a_{i+1}$ for all $i$, $1 \leq i \leq k-1$. Let $f(n)$ be the number of $n$-chains where $n \geq 2$. For example, $f(4) = 2$ corresponds to the 4-chains $\{1, 4\}$ and $\{1, 2, 4\}$. Prove that $f(2^m \cdot 3) = 2^{m-1}(m + 2)$ for every positive integer $m$.

**Walkthrough** —

**(a)** Let us determine $f(2), f(4), f(8), \ldots$.

- Note that $f(2) = 1$ since $\{1, 2\}$ is the only 2-chain.
- Note that $f(2^2) = 2$ since $\{1, 2^2\}$, $\{1, 2, 2^2\}$ are the only $2^2$-chains.
- Note that $f(2^3) = 2^2$ since $\{1, 2^3\}$, $\{1, 2, 2^3\}$, $\{1, 2^2, 2^3\}$, $\{1, 2, 2^2, 2^3\}$ are the only $2^3$-chains.

The above examples suggest that $f(2^m) = 2^{m-1}$ for any integer $m \geq 1$. However, this does require a proof, which we will do in a while.

**(b)** Let us determine $f(2 \cdot 3), f(2^2 \cdot 3), f(2^3 \cdot 3), \ldots$.

- Note that $f(2 \cdot 3) = 3$ since there are only 3 many $2 \cdot 3$-chains.

  Here is the $2 \cdot 3$-chain containing **3** as the smallest multiple of 3.
  - $\{1, \mathbf{3}, 2 \cdot 3\}$,

  Here are the $2 \cdot 3$-chains containing $\mathbf{2 \cdot 3}$ as the smallest multiple of 3.
  - $\{1, 2, \mathbf{2 \cdot 3}\}$,
  - $\{1, \mathbf{2 \cdot 3}\}$
- Note that $f(2^2 \cdot 3) = 2^1 \cdot (2 + 2)$ since there are $2^1 \cdot (2 + 2)$ many $2^2 \cdot 3$-chains.

  Here are the $2^2 \cdot 3$-chains containing **3** as the smallest multiple of 3.
  - $\{1, \mathbf{3}, 2^2 \cdot 3\}$,
  - $\{1, \mathbf{3}, 2 \cdot 3, 2^2 \cdot 3\}$,

  Here are the $2^2 \cdot 3$-chains containing $\mathbf{2 \cdot 3}$ as the smallest multiple of 3.
  - $\{1, 2, \mathbf{2 \cdot 3}, 2^2 \cdot 3\}$,
  - $\{1, \mathbf{2 \cdot 3}, 2^2 \cdot 3\}$,

  Here are the $2^2 \cdot 3$-chains containing $\mathbf{2^2 \cdot 3}$ as the smallest multiple of 3.
  - $\{1, 2^2, \mathbf{2^2 \cdot 3}\}$,
  - $\{1, 2, 2^2, \mathbf{2^2 \cdot 3}\}$,
  - $\{1, \mathbf{2^2 \cdot 3}\}$,
  - $\{1, 2, \mathbf{2^2 \cdot 3}\}$,
- Note that $f(2^3 \cdot 3) = 2^2 \cdot (3 + 2)$ since there are $2^2 \cdot (3 + 2)$ many $2^3 \cdot 3$-chains.

  Here are the $2^3 \cdot 3$-chains containing **3** as the smallest multiple of 3.
  - $\{1, \mathbf{3}, 2^3 \cdot 3\}$,
  - $\{1, \mathbf{3}, 2 \cdot 3, 2^3 \cdot 3\}$,
  - $\{1, \mathbf{3}, 2^2 \cdot 3, 2^3 \cdot 3\}$,
  - $\{1, \mathbf{3}, 2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3\}$,

Here are the $2^3 \cdot 3$-chains containing $\mathbf{2 \cdot 3}$ as the smallest multiple of 3.

- $\{1, 2, \mathbf{2 \cdot 3}, 2^3 \cdot 3\}$,
- $\{1, 2, \mathbf{2 \cdot 3}, 2^2 \cdot 3, 2^3 \cdot 3\}$,
- $\{1, \mathbf{2 \cdot 3}, 2^3 \cdot 3\}$,
- $\{1, \mathbf{2 \cdot 3}, 2^2 \cdot 3, 2^3 \cdot 3\}$,

Here are the $2^3 \cdot 3$-chains containing $\mathbf{2^2 \cdot 3}$ as the smallest multiple of 3.

- $\{1, 2^2, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\}$,
- $\{1, 2, 2^2, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\}$,
- $\{1, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\}$,
- $\{1, 2, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\}$,

Here are the $2^3 \cdot 3$-chains containing $\mathbf{2^3 \cdot 3}$ as the smallest multiple of 3.

- $\{1, 2^3, \mathbf{2^3 \cdot 3}\}$,
- $\{1, 2, 2^3, \mathbf{2^3 \cdot 3}\}$,
- $\{1, 2^2, 2^3, \mathbf{2^3 \cdot 3}\}$,
- $\{1, 2, 2^2, 2^3, \mathbf{2^3 \cdot 3}\}$,
- $\{1, \mathbf{2^3 \cdot 3}\}$,
- $\{1, 2, \mathbf{2^3 \cdot 3}\}$,
- $\{1, 2^2, \mathbf{2^3 \cdot 3}\}$,
- $\{1, 2, 2^2, \mathbf{2^3 \cdot 3}\}$,

The above examples suggest that $f(2^m \cdot 3) = 2^{m-1}(m+2)$ for any integer $m \geq 1$. Certainly, this does require a proof, which we will do in a while.

**Solution 4.**

**Claim —** For any integer $m \geq 1$, there are precisely $2^{m-1}$ many $2^m$-chains. In other words, $f(2^m) = 2^{m-1}$ for any integer $m \geq 1$.

*Proof of the Claim.* In a $2^m$-chain, the smallest term is 1 and the largest term is $2^m$. The remaining terms are some (or none if the chain has only two terms) of the powers of 2 lying between 1 and $2^m$. Thus, to form a $2^m$-chain, one needs to determine the possibilies for the terms other than the smallest and the largest one. To do so, from the remaining $m-1$ many powers of 2 lying between 1 and $2^m$, we can choose as many and arrange them in an increasing order. By the multiplication principle, this can be done in $2^{m-1}$ ways. This proves the Claim. □

**Claim —** Let $m$ be a positive integer, and $0 \leq k < m$ be an integer. The number of $2^m \cdot 3$-chains, which contain $2^k \cdot 3$ as the smallest multiple of 3, is $2^{m-1}$.

*Proof of the Claim.* If $k = 0$, then the $2^m \cdot 3$-chains are in one-to-one correspondence with the $2^m$-chains, and hence, it follows that $f(2^m \cdot 3) = f(2^m) = 2^{m-1}$.

Let us assume that $1 \leq k < m$. Then it is clear that

$$f(2^m \cdot 3) = f(2^k)f(2^{m-k}) + f(2^k)f(2^{m-k}) = 2f(2^k)f(2^{m-k}).$$

Using the first Claim, it follows that

$$f(2^m \cdot 3) = 2^{m-1}.$$

This completes the proof. □

**Claim —** Let $m$ be a positive integer. The number of $2^m \cdot 3$-chains, which contain $2^m \cdot 3$ as the smallest multiple of 3, is $2^m$.

*Proof.* Note that the $2^m \cdot 3$-chains, which contain $2^m \cdot 3$ as the smallest multiple of 3 and also contain $2^m$, are in one-to-one correspondence with the $2^m$-chains. Moreover, the $2^m \cdot 3$-chains, which contain $2^m \cdot 3$ as the smallest multiple of 3 and does not contain $2^m$, are also in one-to-one correspondence with the $2^m$-chains. It follows that the number of $2^m \cdot 3$-chains, which contain $2^m \cdot 3$ as the smallest multiple of 3, is $2 \cdot f(2^m) = 2^m$. □

Combining the two Claims above, it follows that

$$f(2^m \cdot 3) = \sum_{0 \leq k < m} 2^{m-1} + 2^m = 2^{m-1}(m+2).$$

This completes the proof. ■