RMO 2024a

KV or JNV

Exercise 1 (RMO 2024a P1,). Find all positive integers x, y such that $202x + 4x^2 = y^2$.

Solution 1. Let x, y be positive integers satisfying $202x + 4x^2 = y^2$. Note that y^2 is even, and hence, so is y. This shows that 4 divides $y^2 - 4x^2 = 202x$. It follows that x is even.

Note that

$$4y^2 = (4x)^2 + 2 \cdot 101 \cdot (4x)$$

holds, which gives

$$101^2 = (4x + 101 - 2y)(4x + 101 + 2y). \tag{1}$$

Since x, y are positive integers, it follows that $4x + 101 - 2y < 4x + 101 + 2y$. This shows that

$$4x + 101 - 2y = 1, \quad 4x + 101 + 2y = 101^2, \tag{2}$$

which yields

$$x = 1250, \quad y = 2550.$$

Also note that if $x = 1250$ and $y = 2550$, then Eq. (2) holds, which yields Eq. (1), which implies $202x + 4x^2 = y^2$. Hence, the solution to the given equation in the positive integers is precisely

$$x = 1250, \quad y = 2550.$$

■

Exercise 2 (RMO 2024a P2,). Show that there do not exist non-zero real numbers a, b, c such that the following statements hold simultaneously:

1. the equation $ax^2 + bx + c = 0$ has two distinct roots x_1, x_2;
2. the equation $bx^2 + cx + a = 0$ has two distinct roots x_2, x_3;
3. the equation $cx^2 + ax + b = 0$ has two distinct roots x_3, x_1.

(Note that x_1, x_2, x_3 may be real or complex numbers.)

Solution 2. On the contrary, let us assume that there exist nonzero real numbers a, b, c such that the given condition holds. Note that

$$ax_2^2 + bx_2 + c = 0, bx_2^2 + cx_2 + a = 0$$

hold. Eliminating x_2^2, we obtain

$$(b^2 - ca)x_2 = a^2 - bc.$$

Also note that

$$(a^2 - bc)x_2^2 = (c^2 - ab)x_2$$

holds. Since $c \neq 0$, it follows that $x_2 \neq 0$, and hence

$$(a^2 - bc)x_2 = c^2 - ab.$$

Combining the above, we obtain

$$(a^2 - bc)^2 = (b^2 - ca)(c^2 - ab),$$

which can be simplified to

$$a(a^3 + b^3 + c^3 - 3abc) = 0.$$

Since a is nonzero, it implies that $a^3 + b^3 + c^3 = 3abc$, and hence $a + b + c = 0$ or $a = b = c$.

Note that if a, b, c satisfy $a = b = c$, then each of x_1, x_2, x_3 is equal to one of ω, ω^2, and hence, two of x_1, x_2, x_3 are equal, which is impossible by our assumption.

Suppose a, b, c satisfy $a + b + c = 0$. Note that if some two of a, b, c are equal, then one of the polynomials $ax^2 + bx + c, bx^2 + cx + a, cx^2 + ax + b$ does not have distinct roots. Note that if the roots of $ax^2 + bx + c$ are equal, then $1 = -b/a - 1$ holds, which implies that $b = -2a$, and hence $a = c$, which is impossible. Hence, the roots of $ax^2 + bx + c$ are distinct. By similar arguments, it follows that each of $bx^2 + cx + a, cx^2 + ax + b$ possesses distinct roots. If the common root of $ax^2 + bx + c, bx^2 + cx + a$ is equal to 1, then considering the common root of $bx^2 + cx + a, cx^2 + ax + b$, we obtain $c/b = a/c$, which gives $c^2 = ab$, and this yields $(a + b)^2 = ab$, which is impossible since a, b are nonzero real numbers. If the common root of $ax^2 + bx + c, bx^2 + cx + a$ is not equal to 1, then we obtain $b/a = c/b$, which implies that $(c + a)^2 = b^2 = ac$, which is impossible since a, c are nonzero real numbers.

This completes the proof. ■

Exercise 3 (RMO 2024a P4,). Let $n > 1$ be a positive integer. Call a rearrangement $a_1, a_2, \ldots, a_n$ of $1, 2, \ldots, n$ *nice* if for every $k = 2, 3, \ldots, n$, we have that $a_1^2 + a_2^2 + \cdots + a_k^2$ is not divisible by k. Determine which positive integers $n > 1$ have a nice arrangement.

Solution 3. If n is an odd positive integer and is not divisible by 3, then the sum of the squares of $1, 2, \ldots, n$, that is, the integer

$$\frac{n(n+1)(2n+1)}{6}$$

is a multiple of n. This shows that if $n > 1$ is an integer relatively prime to 6, then $1, 2, \ldots, n$ do not admit any nice rearrangement.

Claim — For any positive integer k, the following rearrangement of $1, 2, \ldots, 6k$ (to be read accross the rows)

$$\begin{array}{cccccc} 2 & 1 & 3 & 4 & 6 & 5 \\ 8 & 7 & 9 & 10 & 12 & 11 \\ 14 & 13 & 15 & 16 & 18 & 17 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 6k-4 & 6k-5 & 6k-3 & 6k-2 & 6k & 6k-1 \end{array}$$

is nice.

Proof of the Claim. Note that

$$\begin{aligned} \frac{2^2+1^2}{2} &= \frac{(2+1)(2\cdot 2+1)}{6} \\ \frac{2^2+1^2+3^2}{3} &= \frac{(3+1)(2\cdot 3+1)}{6} \\ \frac{2^2+1^2+3^2+4^2}{4} &= \frac{(4+1)(2\cdot 4+1)}{6} \\ \frac{2^2+1^2+3^2+4^2+6^2}{5} &= \frac{(5+1)(2\cdot 5+1)}{6} + \frac{6^2-5^2}{5} \\ \frac{2^2+1^2+3^2+4^2+6^2+5^2}{6} &= \frac{(6+1)(2\cdot 6+1)}{6} \end{aligned}$$

are not integers. This proves the Claim for $k = 1$. Let k be a positive integer such that the Claim holds. Consider the rearrangement (to be read accross the rows)

$$\begin{array}{cccccc} 2 & 1 & 3 & 4 & 6 & 5 \\ 8 & 7 & 9 & 10 & 12 & 11 \\ 14 & 13 & 15 & 16 & 18 & 17 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 6k-4 & 6k-5 & 6k-3 & 6k-2 & 6k & 6k-1 \\ 6k+2 & 6k+1 & 6k+3 & 6k+4 & 6k+6 & 6k+5 \end{array}$$

of $1, 2, \ldots, 6k+6$. Note that

$$\frac{\sum_{i=1}^{6k} i^2 + (6k+2)^2}{6k+1} = \frac{6k(12k+1)}{6} + \frac{(6k+2)^2}{6k+1}$$

is not an integer since the integers $6k+1, 6k+2$ are relatively prime and $6k+1$ is larger than 1. Observe that

$$\frac{\sum_{i=1}^{6k} i^2 + (6k+2)^2 + (6k+1)^2}{6k+2} = \frac{(6k+3)(12k+5)}{6}$$

is not an integer since the integers $6k+3, 12k+5$ are odd. Also note that

$$\frac{\sum_{i=1}^{6k} i^2 + (6k+2)^2 + (6k+1)^2 + (6k+3)^2}{6k+3} = \frac{(6k+4)(12k+7)}{6}$$

since 3 does not divide any of $6k+4, 12k+7$. Notice that

$$\frac{\sum_{i=1}^{6k} i^2 + (6k+2)^2 + (6k+1)^2 + (6k+3)^2 + (6k+4)^2}{6k+4} = \frac{(6k+5)(12k+9)}{6}$$

is not an integer since $6k+5, 12k+9$ are odd integers. Observe that

$$\frac{\sum_{i=1}^{6k} i^2 + (6k+2)^2 + (6k+1)^2 + (6k+3)^2 + (6k+4)^2 + (6k+6)^2}{6k+5}$$
$$= \frac{(6k+6)(12k+11)}{6} + \frac{(6k+6)^2 - (6k+5)^2}{6k+5}$$

is not an integer since the integers $6k+5, 6k+6$ are relatively prime and $6k+5$ is larger than 1. Also note that

$$\frac{\sum_{i=1}^{6k} i^2 + (6k+2)^2 + (6k+1)^2 + (6k+3)^2 + (6k+4)^2 + (6k+6)^2 + (6k+5)^2}{6k+6}$$
$$= \frac{(6k+7)(12k+13)}{6}$$

is not an integer since the integers $6k+7, 12k+13$ are relatively prime to 6. By induction, the Claim holds for any positive integer k. □

By the above Claim, it follows that if n is a positive integer and n is not relatively prime to 6, then the integers $1, 2, \ldots, n$ admit a nice rearrangement.

This proves that the positive integers $n > 1$, admitting a nice rearrangement, are precisely those which are not relatively prime to 6. ■

Exercise 4 (RMO 2024a P6,). Let X be a set of 11 integers. Prove that one can find a nonempty subset $\{a_1, a_2, \ldots, a_k\}$ of X such that 3 divides k and 9 divides the sum $\sum_{i=1}^{k} 4^i a_i$.

> **Walkthrough** — Use repeatedly the fact that given five integers, the sum of some three of them is divisible by 3.

The content posted here and at this blog by Evan Chen are quite useful.

Solution 4. From the eleven elements of X, one can find three elements whose sum is divisible by 3. From the remaining 8 elements, one can find three elements whose sum is divisible by 3. From the remaining 5 elements, one can find three elements whose sum is divisible by 3. Hence, there are nine elements in X, denoted by $a_1, a_2, \ldots, a_9$ such that the sums

$$a_1 + a_2 + a_3, a_4 + a_5 + a_6, a_7 + a_8 + a_9$$

are divisible by 3. It follows that the sums

$$\alpha := \sum_{i=1}^{3} 4^i a_i, \beta := \sum_{i=4}^{6} 4^i a_i, \gamma := \sum_{i=7}^{9} 4^i a_i$$

are divisible by 3. Each of α, β, γ is congruent to one of $0, 3, 6$ modulo 9. To conclude, let us consider the following cases.

Suppose one of α, β, γ is divisible by 9. Noting that $4^3 \equiv 1 \pmod 9$, we can find a subset of X of size 3 with the desired property. Henceforth, let us assume that none of α, β, γ is divisible by 9.

If all of α, β, γ are congruent to 3 modulo 9, then we can find a subset of X of size 9 with the desired property.

If one of α, β, γ is congruent to 6 modulo 9, and the remaining two are congruent to 3 modulo 9, then we can find a subset of X of size 6 with the desired property.

If one of α, β, γ is congruent to 3 modulo 9, and the remaining two are congruent to 6 modulo 9, then we can find a subset of X of size 6 with the desired property.

If all of α, β, γ are congruent to 6 modulo 9, then we can find a subset of X of size 9 with the desired property.

The completes the proof. ■