RMO 2023b

Non-KV and Non-JNV

Exercise 1 (RMO 2023b P1,). Let $\mathbb{N}$ be the set of all positive integers and

$$S = \left\{(a, b, c, d) \in \mathbb{N}^4 : a^2 + b^2 + c^2 = d^2\right\}.$$

Find the largest positive integer m such that m divides $abcd$ for all $(a, b, c, d) \in S$.

> **Walkthrough** —
>
> **(a)** Show that $(1, 2, 2, 3)$ lies in S, and deduce that m divides 12.
>
> **(b)** Let (a, b, c, d) be an element of S. Show that at least one of a, b, c, d is divisible by 3, and at least one of them is even.
>
> **(c)** Prove that if d is even, then at least one of a, b, c is even, and that if d is odd, then at least two of a, b, c are even.
>
> **(d)** Conclude that m is divisible by $2 \cdot 2 \cdot 3$.

Solution 1. Let m denote the largest positive integer such that it divides $abcd$ for all (a, b, c, d) in S. Note that $1^2 + 2^2 + 2^2 = 3^2$ holds, which shows that $(1, 2, 2, 3)$ lies in S. This shows that m divides 12.

Let (a, b, c, d) be an element of S. Note that at least one of a, b, c, d is divisible by 3 since

$$(\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 \not\equiv (\pm 1)^2 \bmod 3.$$

Also note that if all of a, b, c, d are odd, then we obtain

$$(\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 \equiv (\pm 1)^2 \bmod 4,$$

which is impossible. It follows that at least one of a, b, c, d is even.

If d is even, then at least one of a, b, c is even because the sum of squares of three odd integers is congruent to 3 modulo 4. If d is odd, then at least one of a, b, c is even. Note that $0 + (\pm 1)^2 + (\pm 1)^2 \not\equiv (\pm 1)^2 \bmod 4$, which shows that if d is odd, then at least two of a, b, c are even. This implies that at least two of a, b, c, d are even. Hence, m is divisible by $2 \cdot 2 \cdot 3 = 12$.

This proves that $m = 12$. ∎

Exercise 2 (RMO 2023b P3,). For any natural number n, expressed in base 10, let $s(n)$ denote the sum of all its digits. Find all natural numbers m and n such that $m < n$ and

$$(s(n))^2 = m \text{ and } (s(m))^2 = n.$$

Solution 2. Let m, n be natural numbers satisfying the given conditions. Note that

$$n^2 \equiv m \bmod 9, m^2 \equiv n \bmod 9,$$

which implies that m is congruent to one of $0, 1$ modulo 3, and so is n, and consequently, it follows that

$$m \equiv n \bmod 3.$$

Suppose m has d digits. Note that $m \geq 10^{d-1}$ and $s(m) \leq 9d$ holds. Consequently, $10^{d-1} > (9d)^2$ would yield

$$n > m \geq 10^{d-1} > (9d)^2 \geq s(m)^2 = n,$$

which is impossible. Observe that $10^{5-1} > (9 \cdot 5)^2$ holds, and if k is a positive integer satisfying $10^{k-1} > (9k)^2$, then

$$\begin{aligned} \frac{10^k}{(9(k+1))^2} &> \frac{10(9k)^2}{(9(k+1))^2} \\ &= \frac{10}{(1+\frac{1}{k})^2} \\ &> 1 \end{aligned}$$

holds. By induction, it follows that $d \leq 4$, that is, m has at most 4 digits.

This gives

$$s(m) \leq 9d \leq 36,$$

and hence, it follows that

$$n = s(m)^2 \leq 36^2 = 1296.$$

This yields

$$s(n) \leq \max\{s(1296), s(1199), s(1099), s(999)\} = \max\{18, 20, 19, 27\} = 27,$$

which gives

$$m = s(n)^2 \leq 27^2 = 729.$$

We get

$$s(m) \leq \max\{s(729), s(719), s(709), s(699)\} = \max\{18, 17, 16, 24\} = 24,$$

which shows that

$$n = s(m)^2 \leq 24^2 = 576.$$

This gives

$$s(n) \leq \max\{s(576), s(569), s(559), \ldots, s(519), s(509), s(499)\} = 22,$$

which yields

$$m = s(n)^2 \leq 22^2 = 484.$$

This shows that

$$s(m) \leq \max\{s(484), s(479), s(399)\} = 21,$$

which gives

$$n = s(m)^2 \leq 21^2 = 441.$$

It follows that

$$s(n) \leq \max\{s(441), s(400), s(361), s(324), s(289)\} = \max\{9, 4, 10, 9, 19\} = 19,$$

and hence, we get

$$m = s(n)^2 \leq 19^2 = 361.$$

This gives

$$s(m) \leq \max\{s(361), s(324), s(289)\} = 19,$$

which yields

$$n \leq 19^2,$$

and using the inequality $m < n$, the congruence $m \equiv n \bmod 3$, and that m is a perfect square, we obtain

$$m \leq 17^2.$$

Note that

$$s(17^2)^2 = s(289)^2 = 19^2 = 361, \quad s(361)^2 = 100 \neq 17^2$$

holds, which implies that

$$m \leq 16^2,$$

and hence, we obtain

$$s(m) \leq \max\{s(256), s(225), s(196), s(169), s(81), s(64), s(49)\} = 16.$$

Consequently, we obtain

$$n \leq 16^2,$$

and using the inequality $m < n$, the congruence $m \equiv n \bmod 3$, and that m is a perfect square, we get

$$m \leq 14^2.$$

Observe that

$$s(14^2)^2 = s(196)^2 = 16^2 = 256, \quad s(256)^2 = 13^2 \neq 14^2$$

holds, which yields

$$m \leq 13^2.$$

Note that $(m, n) = (13^2, 16^2)$ is a solution to the given equation. If $8^2 \leq m \leq 12^2$, then

$$s(m) \leq \max\{s(64), s(81), s(100), s(121), s(144)\} = 9,$$

which gives $n \leq 81$, and hence (m, n) is equal to $(8^2, 9^2)$, which is impossible. It follows that $m \leq 7^2$. Observe that $s(m)^2 = n, s(n)^2 = m$ fails to hold if m is any of $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2$.

We conclude that $(m, n) = (13^2, 16^2)$ is the only solution to the given equations. ■

Remark. The above solution to Exercise 2 makes an effort to defer computing the (decimal expansions of the) perfect squares. It turns out that without deferring it, we may arrive at the solution in fewer steps.

After obtaining $n \leq 36^2 = 1296$, one may compute the sum of the digits of the perfect squares lying between 11^2 and 36^2, as recorded in the table below.

k	k^2	$s(k^2)$
36	1296	18
35	1225	10
34	1156	13
33	1089	18
32	1024	7
31	961	16
30	900	9
29	841	13
28	784	19
27	729	18
26	676	19
25	625	13
24	576	18
23	529	16
22	484	16
21	441	9
20	400	4
19	361	10
18	324	9
17	289	19
16	256	13
15	225	9

14	196	16
13	169	16
12	144	9
11	121	4

This yields

$$s(n) \leq 19,$$

which gives

$$m = s(n)^2 \leq 19^2 = 361.$$

We get

$$s(m) \leq 19,$$

which shows that

$$n = s(m)^2 \leq 19^2 = 361.$$

Since $m \equiv n \bmod 3$, we obtain

$$m \leq 17^2 = 289.$$

Note that

$$s(17^2)^2 = s(289)^2 = 19^2 = 361, \quad s(361)^2 = 100 \neq 17^2$$

holds, which implies that

$$m \leq 16^2,$$

and hence, we obtain

$$s(m) \leq \max\{s(256), s(225), s(196), s(169), s(81), s(64), s(49)\} = 16.$$

Consequently, we obtain

$$n \leq 16^2,$$

and this gives

$$m \leq 15^2.$$

Observe that

$$s(15^2)^2 = s(225)^2 = 81, \quad s(81)^2 = 81 \neq 15^2,$$

$$s(14^2)^2 = s(196)^2 = 16^2 = 256, \quad s(256)^2 = 13^2 \neq 14^2$$

holds, which yields

$$m \leq 13^2.$$

Note that $(m, n) = (13^2, 16^2)$ is a solution to the given equation. If $8^2 \leq m \leq 12^2$, then

$$s(m) \leq \max\{s(64), s(81), s(100), s(121), s(144)\} = 9,$$

which gives $n \leq 81$, and hence (m, n) is equal to $(8^2, 9^2)$, which is impossible. It follows that $m \leq 7^2$. Observe that $s(m)^2 = n, s(n)^2 = m$ fails to hold if m is any of $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2$.

We conclude that $(m, n) = (13^2, 16^2)$ is the only solution to the given equations.

Exercise 3 (RMO 2023b P5,). Let $n > k > 1$ be positive integers.

Determine all positive real numbers $a_1, a_2, \ldots, a_n$ which satisfy

$$\sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} = \sum_{i=1}^{n} a_i = n.$$

Solution 3. Let $a_1, \ldots, a_n$ be positive reals satisfying the above condition. Note that

$$\begin{aligned}
&\sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}} \\
&\leq \sum_{i=1}^{n} \sqrt{\frac{a_i^k}{\sqrt[k]{a_i^{k(k-1)}}}} \quad \text{(by the AM-GM inequality)} \\
&= \sum_{i=1}^{n} \sqrt{a_i} \\
&\leq \left(n \sum_{i=1}^{n} a_i \right)^{1/2} \quad \text{(by the Cauchy–Schwarz inequality)} \\
&= \sum_{i=1}^{n} \sqrt{\frac{ka_i^k}{(k-1)a_i^k + 1}}
\end{aligned}$$

hold, and hence, the intermediate inequalities are equalities. This implies that $a_1, a_2, \ldots, a_n$ are equal. Using the given condition, it follows that

$$a_1 = a_2 = \cdots = a_n = 1.$$

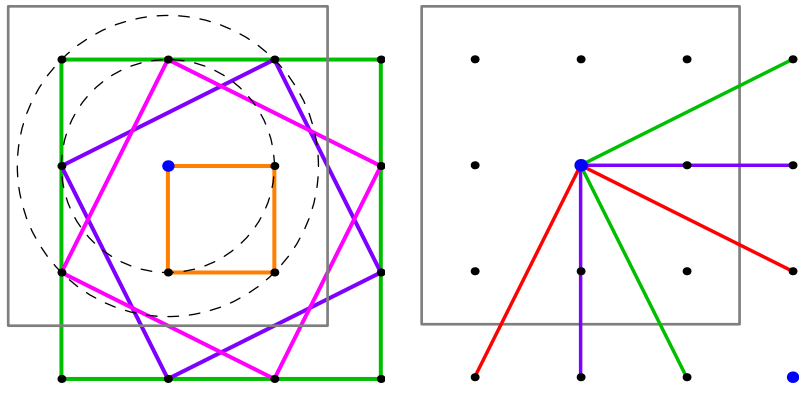
Also note that the given conditions are satisfied if $a_1 = a_2 = \cdots = a_n = 1$ holds. This proves that the positive real numbers satisfying the given condition are precisely

$$a_1 = a_2 = \cdots = a_n = 1.$$

■

Exercise 4 (RMO 2023b P6,). Consider a set of 16 points arranged in a 4×4 square grid formation. Prove that if any 7 of these points are coloured blue, then there exists an isosceles right-angled triangle whose vertices are all blue.

The following is from AoPS, is due to Rohan (Goyal?) as mentioned here by L567.



(a) (b)

Figure 1: RMO 2023b P6, Exercise 4

Walkthrough —

(a) Show that if the small square (as in Fig. 1a) does not contain a blue point, then we are done.

(b) It remains to consider the case when the small square (as in Fig. 1a) contains at least one blue point.

(c) Rotating the configuration about the center of the small square (if necessary), assume that the top-left vertex of the small square (as in Fig. 1a) is blue.

(d) Prove that the gray square contains at most three blue points.

- Consider the case when at least two blue points lie on the bigger dashed circle. Show that the smaller dashed circle does not contain any blue point, in this case. Hence, the gray square contains at most three blue points.
- Similarly, if the smaller dashed circle contains at least two blue points, then the gray square (as in Fig. 1b) contains at most three blue points.

(e) It suffices to consider that each one of the red, purple, and green L-shapes, has at most one end-point which is blue (otherwise, we are done).

(f) Use the above to show that the bottom-right point (as in Fig. 1b) is blue.

(g) Consider the point at the bottom-right corner, and the center of the gray square, and a blue end-point of an L-shape, to show that these three points form the vertices of an isosceles triangle having the required properties.

Solution 4. Note that the 16 points are the vertices of the four squares as in Fig. 1a. If no vertex of the small square is blue, then by the pigeonhole principle, at least one of the remaining three squares has at least three blue

vertices, and hence there exists an isosceles right-angled triangle with blue vertices.

Let us assume that the small square (as in Fig. 1a) has a blue vertex. Rotating the configuration about the center of the small square (if necessary), we may and do assume that the top-left vertex of the small square is blue. Henceforth, on the contrary, let us assume that there are no isosceles right-angled triangles with blue vertices.

Claim — The gray square (as in Fig. 1a) contains at most three blue points.

Proof of the Claim. Note that the points within the gray square lies on the two dashed circles (as in Fig. 1a). Therefore, to prove the Claim, it suffices to show that if one of the dashed circles (as in Fig. 1a) contains at least two blue points, then the other dashed circle does not contain any blue point.

If at least two blue points lie on the bigger dashed circle, then using the assumption, it follows that no more blue points lies on it, and hence these two blue points lie along a diameter. It follows that no blue point lies on the other dashed circle.

If at least two blue points lie on the smaller dashed circle, then using a similar argument, it follows that no blue point lies on the bigger dashed circle.

The Claim follows. □

Note that if both the end-points of one of the red, purple, and green L-shapes (as in Fig. 1b) are blue, then these points together with the center of gray square form the vertices of an isosceles right-angled triangle, contradicting the assumption. Hence, each one of these three L-shapes has at most one end-point which is blue. Since there are 7 blue points, using the Claim, it follows that the bottom-right point is blue. Note that the center of the gray square, the bottom-right point, and a blue end-point of the "purple" L-shape, form the vertices of a triangle having the required properties. ■