RMO 2023a

KV or JNV

Exercise 1 (RMO 2023a P2,). Given a prime number p such that $2p$ is equal to the sum of the squares of some four consecutive positive integers. Prove that $p-7$ is divisible by 36.

> **Walkthrough** — Show that the sum of four consecutive squares is congruent to 6 modulo 8, and conclude that $p \equiv 3 \bmod 4$. Considering congruence conditions modulo 3, prove that the smallest of the four consecutive numbers is a multiple of 3. Deduce that the sum of the four consecutive squares is 5 modulo 9.

Solution 1. Let p be a prime satisfying the given condition. Write

$$2p = x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2$$

where x is a positive integer. Note that

$$\begin{aligned} x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2 &= 4x^2 + 12x + 14 \\ &\equiv 4x(x+1) + 6 \bmod 8 \\ &\equiv 6 \bmod 8 \end{aligned}$$

holds. It follows that $2p$ is congruent to 6 modulo 8, which shows that p is congruent to 3 modulo 4.

Note that if $x \equiv \pm 1 \bmod 3$, then

$$\begin{aligned} x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2 &= 4x^2 + 12x + 14 \\ &\equiv x^2 + 2 \bmod 3 \\ &\equiv 0 \bmod 3 \end{aligned}$$

holds, which shows that 3 divides $2p$, and hence $p = 3$. However, 6 cannot be expressed as the sum of four consecutive positive integers since $3^2 > 6$. This shows that x is divisible by 3. It follows that

$$\begin{aligned} x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2 &= 4x^2 + 12x + 14 \\ &\equiv 14 \bmod 9. \end{aligned}$$

So, $2p - 14$ is a multiple of 9, and hence, it is an even multiple of 9, implying that $p - 7$ is a multiple of 9. Since $p \equiv 3 \bmod 4$, we obtain $p \equiv 7 \bmod 36$. ■

Exercise 2 (RMO 2023a P3,). Let $f(x)$ be a polynomial with real coefficients of degree 2. Suppose that for some pairwise distinct **nonzero** real numbers , a, b, c we have:

$$f(a) = bc, f(b) = ac, f(c) = ab.$$

Dertermine $f(a + b + c)$ in terms of a, b, c.

Solution 2. Note that cubic polynomial $xf(x) - abc$ has three roots a, b, c, and hence

$$xf(x) - abc = \lambda(x - a)(x - b)(x - c)$$

holds for some real number λ. Substituting $x = 0$ in the above, and using that a, b, c are nonzero, it follows that $\lambda = 1$. This gives $f(x) = x^2 - (a + b + c)x + ab + bc + ca$. We obtain that

$$f(a + b + c) = ab + bc + ca.$$

■

Exercise 3 (RMO 2023a P4,). The set X of N four-digit numbers formed from the digits $1, 2, 3, 4, 5, 6, 7, 8$ satisfies the following condition: *for any two different digits from* $1, 2, 3, 4, 5, 6, 7, 8$*, there exists a number in* X *which contains both of them.* Determine the smallest possible value of N.

First, let's work on it. Let X be a set satisfying the required conditions and it has the minimum cardinality among such sets. Given an element x of X, we may assume that x has distinct digits. Otherwise, any repetition of a digit can be replaced by an integer from $\{1, \ldots, 8\}$ which does not occur as a digit in x. Note that after this modification of an element of X, the modified set has the same cardinality as the initial set, and it continues to have the property that for any two different digits from $\{1, \ldots, 8\}$, there exists a number in the modified set which contains both of them. Thus, modifying the elements of X if required, we may assume that the digits of the elements of X are distinct. By the hypothesis on the cardinality of X, it follows that no two elements of X are equal up to permutation. So, the elements of X can be considered as subsets of $\{1, 2, \ldots, 8\}$ of size 4, and X can be thought as a set of certain size 4 subsets of $\{1, \ldots, 8\}$ such that any size two subsets of $\{1, \ldots, 8\}$ is contained in some element of X.

Consider the set

$$A := \{(P, Q) \,|\, |P| = 2, Q \in X, P \subseteq Q\}.$$

Note that the size of A is $\binom{4}{2}|X| = 6|X|$. By hypothesis, it follows that A contains at least $\binom{8}{2} = 28$ elements. Hence, X contains at least 5 elements.

With some effort, one can find the following set

$$\{\{1,2,3,4\},\{5,6,7,8\},\{1,5,2,6\},\{3,7,4,8\},\{1,7,8,2\},\{3,5,6,4\}\}.$$

So far, we have proved that X contains at least 5 elements. Based on this information and the above example only (possibly together with the fact that we are not able to come up with a set consisting of five size 4 subsets of $\{1,\dots,8\}$ having the stated property), we are not in a position to conclude that X has cardinality 6.

In fact, a modification of the above argument does prove that $|X| \geq 6$. Indeed, consider the set

$$A' := \{(P,Q) \,|\, |P| = 1, Q \in X, P \subseteq Q\}.$$

Note that the size of A' is $4|X|$. Also note that for any element $i \in \{1,\dots,8\}$, it is contained in at least three elements of X. Indeed, there are 7 size two subsets of $\{1,\dots,8\}$ containing i, and the union of no two size 4 subsets of $\{1,\dots,8\}$ contains all the size two subsets of $\{1,\dots,8\}$ containing i. By hypothesis, it follows that A' contains at least $8 \cdot 3 = 24$ elements. Hence, X contains at least 6 elements.

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Note that the set

$$\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{1, 5, 2, 6\}, \{3, 7, 4, 8\}, \{1, 7, 8, 2\}, \{3, 5, 6, 4\}\}$$

has size 6 and it consists of certain size 4 subsets of $\{1, \ldots, 8\}$ such that any size two subset of $\{1, \ldots, 8\}$ is contained in one such size 4 subset.

This proves that the smallest possible value of N is 6. ■

Solution 4. Note that the following set of four-digit numbers satisfy the required condition.

$$\{1234, 5678, 1256, 3478, 1278, 3456\}$$

This shows that the smallest possible value of N is at most 6.

Let X be a set of N four-digit numbers satisfying the given condition. Note that for any integer $1 \leq i \leq 8$, two four-digit numbers containing i as a digit, together contains at most 7 distinct elements of $\{1, 2, \ldots, 8\}$ as digits. Hence, for any $1 \leq i \leq 8$, there are at least three elements in X which contain i as a digit. Counting the number of pairs of the form (i, x), where $1 \leq i \leq 8$ and x is an element of X containing i as a digit, we obtain

$$4N \geq 8 \cdot 3,$$

which yields $N \geq 6$.

This proves that the smallest possible value of N is 6. ■

> **Remark.** Is it helpful to count the pairs of the form $(\{i, j\}, x)$ where $1 \leq i < j \leq 8$ and x is an element of X containing i, j as digits?