RMO 2019b

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Exercise 1 (RMO 2019b P1,). For each $n \in \mathbb{N}$ let d_n denote the greatest common divisor of n and $(2019 - n)$. Find value of $d_1 + d_2 + \cdots + d_{2018} + d_{2019}$.

Solution 1. Note that for a positive integer n, the greatest common divisor of n and $2019 - n$ is equal to the greatest common divisor of n and 2019. Also note that $2019 = 3 \cdot 673$, and that $3, 673$ are primes. This shows that for an integer n satisfying $1 \leq n < 2019$, the greatest common divisor of n and $2019 - n$ is equal to

$$\begin{cases} 1 & \text{if } n \text{ is relatively prime to 3 and to 673,} \\ 3 & \text{if 3 divides } n, \\ 673 & \text{if 673 divides } n. \end{cases}$$

This yields

$$\begin{aligned} & d_1 + d_2 + \cdots + d_{2018} + d_{2019} \\ &= \sum_{\substack{1 \leq n < 2019, \\ \gcd(n, 2019) = 1}} 1 + \sum_{\substack{1 \leq n < 2019, \\ 3 \mid n}} 3 + \sum_{\substack{1 \leq n < 2019, \\ 673 \mid n}} 673 + 2019 \\ &= 2019 - \frac{2019}{3} - \frac{2019}{673} + \frac{2019}{3 \cdot 673} + 3 \cdot 672 + 673 \cdot 2 + 3 \cdot 673 \\ &= 2 \cdot 672 + 3 \cdot 672 + 5 \cdot 673 \\ &= 5 \cdot 1345 \\ &= 6725. \end{aligned}$$

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Alternative solution. Note that for a positive integer n, the greatest common divisor of n and $2019 - n$ is equal to the greatest common divisor of n and 2019. Also note that $2019 = 3 \cdot 673$, and that $3, 673$ are primes. It follows that

$$\begin{aligned} & d_1 + d_2 + \cdots + d_{2019} \\ &= \gcd(1, 2019) + \gcd(2, 2019) + \cdots + \gcd(2019, 2019) \end{aligned}$$

$$= (\gcd(1,3) + \gcd(2,3) + \gcd(3,3))\,(\gcd(1,673) + \gcd(2,673) + \cdots + \gcd(673,673))$$
$$= (1+1+3)(1+1+\cdots+1+673)$$
$$= 5 \cdot (672+673)$$
$$= 5 \cdot 1345$$
$$= 6725.$$

□

Exercise 2 (RMO 2019b P3,). Find all triples of non-negative real numbers (a, b, c) which satisfy the following set of equations

$$a^2 + ab = c,$$
$$b^2 + bc = a,$$
$$c^2 + ca = b.$$

Solution 2. Let a, b, c be nonnegative real numbers satisfying the given condition. Note that

$$a^2 - b^2 = (c-a)(1+b), b^2 - c^2 = (a-b)(1+c)$$

hold. If $a > b$ holds, then we get $c > a$ and $b > c$, implying $b > a$. Similarly, if $a < b$ holds, then we obtain $c < a$ and $b < c$, implying $b < a$. This shows that $a = b$, and hence $c = a$. Using $a = b = c$ and the given condition, we obtain $2a^2 = a$, and hence $a = 0$ or $a = 1/2$. This yields $(a, b, c) = (0, 0, 0)$ or $(a, b, c) = (1/2, 1/2, 1/2)$. Note that the tuples $(0, 0, 0), (1/2, 1/2, 1/2)$ satisfy the given conditions. This proves that the required tuples are $(0, 0, 0), (1/2, 1/2, 1/2)$. ■

Exercise 3 (RMO 2019b P4,). Let $a_1, a_2, \ldots, a_6, a_7$ be seven positive integers. Let S be the set of all numbers of the form $a_i^2 + a_j^2$ where $1 \leq i < j \leq 7$. Prove that there exist two elements of S which have the same remainder on dividing by 36.

Solution 3. Note that the square of an integer is congruent to one of $0, 1, 4, 7$ modulo 9. It follows that the sum of two squares is congruent to one of $0, 1, 2, 4, 5, 7, 8$ modulo 9. Denote the set $\{0, 1, 2, 4, 5, 7, 8\}$ by R. Note that the sum of two squares is congruent to one of $0, 1, 2$ modulo 4. Let P denote the set of pairs of the form (i, j) with $1 \leq i < j \leq 7$.

If for some $s \in \{0, 1, 2\}$, there are at least eight pairs (i, j) in P such that the corresponding sums are congruent to s modulo 4, then by the pigeonhole principle, the result follows.

Suppose this does not hold. Noting that $\binom{7}{2} = 21 = 3 \cdot 7$, it follows that for each $s \in \{0, 1, 2\}$, there are precisely seven pairs in P such that the corresponding sums are congruent to s modulo 4.

In particular, there are seven pairs (i, j) in P such that the corresponding sums are divisible by 4, and hence, for any such pair, the integers a_i, a_j are even. If precisely k among the integers $a_1, a_2, \ldots, a_7$ are even, then there are $\binom{k}{2}$ pairs in P such that the corresponding sum is divisible by 4. We obtain $\binom{k}{2} = 7$, which holds for no integer $k \geq 0$. This completes the proof. ■

Exercise 4 (RMO 2019b P5,). There is a pack of 27 distinct cards, and each card has three values on it. The first value is a shape from $\{\Delta, \square, \odot\}$; the second value is a letter from $\{A, B, C\}$; and the third value is a number from $\{1, 2, 3\}$. In how many ways can we choose an unordered set of 3 cards from the pack, so that no two of the chosen cards have two matching values. For example we can chose $\{\Delta A1, \Delta B2, \odot C3\}$. But we cannot choose $\{\Delta A1, \square B2, \Delta C1\}$.

Solution 4. Let $\mathcal{A}$ denote the set of ordered tuples (u, v, w) with $u, v, w \in (\mathbb{Z}/3\mathbb{Z})^3$ such that no two among u, v, w have two equal coordinates. For $u \in (\mathbb{Z}/3\mathbb{Z})^3$, let $\mathcal{A}_u$ denote the set of ordered tuples lying in $\mathcal{A}$ with the first coordinate equal to u. Note that $(u, v, w) \mapsto (u, v, w) - (u, u, u)$ defines a bijection between $\mathcal{A}_u$ and $\mathcal{A}_0$. Hence, the number of ways of choosing an unordered set of 3 cards satisfying the given conditions is $\frac{1}{3!} \cdot 27 \cdot |\mathcal{A}_0|$. Note that

$$\mathcal{A}_0 = \{(0, v, w) \mid v \notin \langle e_1 \rangle \cup \langle e_2 \rangle \cup \langle e_3 \rangle, w \notin \langle e_1 \rangle \cup \langle e_2 \rangle \cup \langle e_3 \rangle, \\ w \notin (v + \langle e_1 \rangle) \cup (v + \langle e_2 \rangle) \cup (v + \langle e_3 \rangle)\},$$

and hence

$$|\mathcal{A}_0| = \sum_{v \notin \langle e_1 \rangle \cup \langle e_2 \rangle \cup \langle e_3 \rangle} \left| (\langle e_1 \rangle \cup \langle e_2 \rangle \cup \langle e_3 \rangle)^c \bigcap ((v + \langle e_1 \rangle) \cup (v + \langle e_2 \rangle) \cup (v + \langle e_3 \rangle))^c \right|.$$

For any subset S of $(\mathbb{Z}/3\mathbb{Z})^3$ and an element $v \in (\mathbb{Z}/3\mathbb{Z})^3$, we have

$$\begin{aligned} |S^c \cap (v + S)^c| &= 27 - |(S^c \cap (v + S)^c)^c| \\ &= 27 - |S \cup (v + S)| \\ &= 27 - |S| - |v + S| + |S \cap (v + S)| \\ &= 27 - 7 - 7 + |S \cap (v + S)| \\ &= 13 + |S \cap (v + S)|. \end{aligned}$$

Henceforth, S denotes the subset $\langle e_1 \rangle \cup \langle e_2 \rangle \cup \langle e_3 \rangle$ of $(\mathbb{Z}/3\mathbb{Z})^3$, and v denotes an element of $(\mathbb{Z}/3\mathbb{Z})^3$ lying outside S.

First, let us consider the case when v has exactly two nonzero coordinates. It follows that v lies in exactly one of $\langle e_1 \rangle - \langle e_2 \rangle$, $\langle e_2 \rangle - \langle e_3 \rangle$, $\langle e_3 \rangle - \langle e_1 \rangle$. Observe that the set $S \cap (v + S)$ is equal to the union of $\langle e_i \rangle \cap (v + \langle e_j \rangle)$ for $1 \leq i, j \leq n$. It follows that $S \cap (v + S)$ has size two.

Now, let us consider the case when all coordinates of v are nonzero. Note that $\langle e_i \rangle \cap (v + \langle e_j \rangle)$ is empty for any $1 \leq i, j \leq n$. Hence, so is the set $S \cap (v + S)$.

Note that the number of elements of $(\mathbb{Z}/3\mathbb{Z})^3$ having exactly two nonzero coordinates is $3 \cdot 2 \cdot 2 = 12$, and the number of elements of $(\mathbb{Z}/3\mathbb{Z})^3$ having all coordinates nonzero is $2^3 = 8$. It follows that

$$|\mathcal{A}_0| = 12 \cdot (13 + 2) + 8 \cdot (13 + 0) = 284.$$

Hence, the number of ways of choosing an unordered set of 3 cards satisfying the given conditions is $\frac{1}{3!} \cdot 27 \cdot 284 = 1278$. ■