RMO 2019a

Exercise 1 (RMO 2019a P1,). Suppose x is a nonzero real number such that both x^5 and $20x + \frac{19}{x}$ are rational numbers. Prove that x is a rational number.

Solution 1. Let us establish the following more general claim.

> **Claim** — Let x be a nonzero real number and r be a rational number such that x^5 and $x + \frac{r}{x}$ are rational numbers. Then x is a rational number.

Proof of the Claim. Note that

$$x^2 + \frac{r^2}{x^2}, x^3 + \frac{r^3}{x^3}$$

are rational. Write $s = x^5$ and note that

$$x^3 + \frac{r^3}{x^3} = \frac{r^3}{s}x^2 + \frac{s}{x^2}.$$

It follows that there are rational numbers a, b such that

$$\begin{pmatrix} 1 & r^2 \\ \frac{r^3}{s} & s \end{pmatrix} \begin{pmatrix} x^2 \\ \frac{1}{x^2} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

holds. If the determinant of $\begin{pmatrix} 1 & r^2 \\ \frac{r^3}{s} & s \end{pmatrix}$ is nonzero, then it follows that x^2 is rational, and using the rationality of x^5, we obtain that x is rational. If the determinant of $\begin{pmatrix} 1 & r^2 \\ \frac{r^3}{s} & s \end{pmatrix}$ is zero, then we get $s^2 = r^5$, which implies that $x^{10} = r^5$, and hence, x^2 is rational, and consequently, so is x. □

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Exercise 2 (RMO 2019a P3,). Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{a}{a^2 + b^3 + c^3} + \frac{b}{b^2 + a^3 + c^3} + \frac{c}{c^2 + a^3 + b^3} \leq \frac{1}{5abc}.$$

Walkthrough — **Homogenize** the denominator and apply the AM-GM inequality.

Solution 2. Note that

$$\begin{aligned}\frac{a}{a^2+b^3+c^3} &= \frac{a}{a^2(a+b+c)+b^3+c^3}\\ &\le \frac{a}{5\sqrt[5]{a^6\cdot abc\cdot b^3\cdot c^3}}\\ &= \frac{1}{5\sqrt[5]{a^2b^4c^4}}\\ &= \frac{1}{5abc}\left(a^3bc\right)^{1/5}\\ &\le \frac{3a+b+c}{25abc}\end{aligned}$$

holds. Similarly, it follows that

$$\begin{aligned}\frac{b}{b^2+a^3+c^3} &\le \frac{3b+a+c}{25abc},\\ \frac{c}{c^2+a^3+b^3} &\le \frac{3c+a+b}{25abc}.\end{aligned}$$

Adding the above inequalities, we obtain

$$\begin{aligned}&\frac{a}{a^2+b^3+c^3}+\frac{b}{b^2+a^3+c^3}+\frac{c}{c^2+a^3+b^3}\\ &\le \frac{3a+b+c}{25abc}+\frac{3b+a+c}{25abc}+\frac{3c+a+b}{25abc}\\ &= \frac{a+b+c}{5abc}\\ &= \frac{1}{5abc}.\end{aligned}$$

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Exercise 3 (RMO 2019a P4,). Consider the following 3×2 array formed by using the numbers $1,2,3,4,5,6$,

$$\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{pmatrix} = \begin{pmatrix}1 & 6\\ 2 & 5\\ 3 & 4\end{pmatrix}.$$

Observe that all row sums are equal, but the sum of the square of the squares is not the same for each row. Extend the above array to a $3\times k$ array $(a_{ij})_{3\times k}$ for a suitable k, adding more columns, using the numbers $7,8,9,\ldots,3k$ such that

$$\sum_{j=1}^{k} a_{1j} = \sum_{j=1}^{k} a_{2j} = \sum_{j=1}^{k} a_{3j} \quad\text{and}\quad \sum_{j=1}^{k} (a_{1j})^2 = \sum_{j=1}^{k} (a_{2j})^2 = \sum_{j=1}^{k} (a_{3j})^2$$

The content posted here and at this blog by Evan Chen are quite useful.

Solution 3. Note that

$$\begin{pmatrix} 1 & 6 & 2+6 & 5+6 & 3+2\cdot 6 & 4+2\cdot 6 \\ 2 & 5 & 3+6 & 4+6 & 1+2\cdot 6 & 6+2\cdot 6 \\ 3 & 4 & 1+6 & 6+6 & 2+2\cdot 6 & 5+2\cdot 6 \end{pmatrix}$$

works. ■

Exercise 4 (RMO 2019a P6,). Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers a, b, c, d among them such that $\gcd(a, b) = \gcd(b, c) = \gcd(c, d) = \gcd(d, a) = 1$.

Solution 4. Denote the chosen 91 integers by $n_1, \ldots, n_{91}$. For $1 \le i \le 91$, let d_i denote the number of integers among them which are relatively prime with n_i. Consider the triples (a, b, c) where a, b, c run over the chosen 91 integers, and satisfy $\gcd(a, b) = \gcd(a, c) = 1$. Counting these triples along their first entry, it follows that the number of such triples is

$$\binom{d_1}{2} + \cdots + \binom{d_{91}}{2}.$$

If the statement to be proved is false, then the given any two distinct integers b, c among them, it can be completed to at most one such triple (a, b, c). This shows that

$$\binom{d_1}{2} + \cdots + \binom{d_{91}}{2} \le \binom{91}{2},$$

which implies that

$$\begin{aligned} \binom{91}{2} &\ge \frac{1}{2}\left(d_1^2 + \cdots + d_{91}^2 - d_1 - \cdots - d_{91}\right) \\ &= \frac{1}{2}\left(d_1^2 + \cdots + d_{91}^2 - 2\cdot 456\right) \\ &\ge \frac{1}{2}\left(\frac{(d_1 + \cdots + d_{91})^2}{91} - 912\right) \quad \text{(using the Cauchy–Schwartz inequality)} \\ &= \frac{1}{2}\left(\frac{(2\cdot 456)^2}{91} - 912\right) \\ &= 456 \cdot 9, \end{aligned}$$

and this yields $910 > 912$, which is impossible. This completes the proof. ■