RMO 2018b

Kerala and Tamil Nadu

Exercise 1 (RMO 2018b P2,). Find the set of all real values of a for which the real polynomial equation $P(x) = x^2 - 2ax + b = 0$ has real roots, given that $P(0) \cdot P(1) \cdot P(2) \neq 0$ and $P(0), P(1), P(2)$ form a geometric progression.

Solution 1. Let a, b be real numbers such that the polynomial $x^2 - 2ax + b$ has real roots, the product $P(0) \cdot P(1) \cdot P(2)$ is nonzero, and $P(0), P(1), P(2)$ form a geometric progression. Since $x^2 - 2ax + b$ has real roots, we get $a^2 \geq b$. Using $P(0), P(1), P(2)$ form a geometric progression, we obtain

$$b(4 - 4a + b) = (1 - 2a + b)^2,$$

which implies

$$4b - 4ab + b^2 = 1 + 4a^2 + b^2 - 4a - 4ab + 2b,$$

which gives $2b = (2a - 1)^2$. Combining the above with $a^2 \geq b$ yields $2a^2 \geq (2a - 1)^2$, which implies that $2a^2 - 4a + 1 \leq 0$, which is equivalent to

$$1 - \frac{1}{\sqrt{2}} \leq a \leq 1 + \frac{1}{\sqrt{2}}.$$

Note that

$$\begin{aligned} P(0) &= \frac{(2a-1)^2}{2}, \\ P(1) &= 1 - 2a + b \\ &= 1 - 2a + \frac{(2a-1)^2}{2} \\ &= (1 - 2a)\left(\frac{3}{2} - a\right), \\ P(2) &= 4 - 4a + \frac{(2a-1)^2}{\sqrt{2}} \\ &= \frac{4a^2 - 12a + 9}{2} \end{aligned}$$

$$= \frac{(2a-3)^2}{2}.$$

This shows that a lies in

$$\left(1-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right) \setminus \left\{\frac{1}{2}, \frac{3}{2}\right\}.$$

Note that for any a lying in the above set, taking $b = (2a-1)^2/2$, it follows that $P(0), P(1), P(2)$ are nonzero, and form a geometric progression. This shows that the desired set is equal to

$$\left(1-\frac{1}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right) \setminus \left\{\frac{1}{2}, \frac{3}{2}\right\}.$$

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Exercise 2 (RMO 2018b P3,). **Show that there are infinitely many tuples (a, b, c, d) of natural numbers such that $a^3 + b^4 + c^5 = d^7$.**

Walkthrough —

(a) Let x, y, z, w be positive integers satisfying $x + y + z = w$.

(b) Multiply the equation $x + y + z = w$ by $x^\alpha y^\beta z^\gamma w^\delta$ where $\alpha, \beta, \gamma, \delta$ are positive integers.

(c) Can one choose $\alpha, \beta, \gamma, \delta$ such that the resulting equation is of the form $a^3 + b^4 + c^5 = d^7$ for some positive integers a, b, c, d?

Exercise 3 (RMO 2018b P4,). **Suppose 100 points in the plane are coloured using two colours, red and white, such that each red point is the centre of a circle passing through at least three white points. What is the least possible number of white points?**

Summary — It relies on the fact that one can find enough points on the plane such that no three of them are collinear and no four of them are concyclic.

Walkthrough —

(a) There is an upper bound on the number of the red points in terms of the number of the white points. This gives an upper bound on the total number of points, which is 100, in terms of the number of the white points.

(b) Use this bound to guess the least possible number of the white points, which would turn out to be 10.

(c) Begin with 10 white points on the plane in *general position*, and then,

The content posted here and at this blog by Evan Chen are quite useful.

introduce enough red points to construct a configuration of 100 points with the stated properties.

Solution 2. Let n denote the number of white points. Since each red point is the centre of a circle passing through at least three white points, it follows that the number of red points is at most $\binom{n}{3}$. This shows that

$$n + \binom{n}{3} \geq 100.$$

Note that $n \mapsto n + \binom{n}{3}$ defines an increasing function on the nonnegative integers. Observe that

$$9 + \binom{9}{3} = 93, \quad 10 + \binom{10}{3} = 130.$$

This implies that $n \geq 10$.

We claim that there is a configuration of 100 points on the plane such that it admits a coloring using two colors, red and white, such that precisely 10 points are colored white, and that each red point is the centre of a circle passing through at least three white points. Indeed, consider 10 points on the plane such that no three of them are collinear and no four of them are concyclic [1]. Color these 10 points white. These white points have $\binom{10}{3} = 120$ subsets of size 3. Consider only 90 such subsets of the white points, and for any such subset of size 3, color the center of the circle passing through them red. Since no three white points are collinear and no four white points are concyclic, it follows that there are precisely 90 pairwise distinct red points. So, the red and the white points together form a set of 100 points such that each red point is the centre of a circle passing through at least three white points. ■

Exercise 4 (RMO 2018b P6,). Define a sequence $\{a_n\}_{n\geq 1}$ of real numbers by

$$a_1 = 2, \qquad a_{n+1} = \frac{a_n^2 + 1}{2}, \text{ for } n \geq 1.$$

Prove that

$$\sum_{j=1}^{N} \frac{1}{a_j + 1} < 1$$

for every natural number N.

Solution 3. Note that

$$2(a_{n+1} - 1) = a_n^2 - 1$$

[1] Why does such a collection exist? This could be intuitively clear, but can you write down a precise proof? Does induction help?

holds for any $n \geq 1$. Also note that $a_1 > 1$, and by induction, it follows that $a_n > 1$ for any $n \geq 2$. This shows that

$$\begin{aligned}\frac{1}{a_n+1} &= \frac{a_n+1-2}{a_n^2-1}\\ &= \frac{1}{a_n-1} - \frac{2}{a_n^2-1}\\ &= \frac{1}{a_n-1} - \frac{1}{a_{n+1}-1}\end{aligned}$$

holds for any $n \geq 1$. Consequently, for any natural number N, we obtain

$$\sum_{j=1}^{N} \frac{1}{a_j+1} = \sum_{j=1}^{N} \left(\frac{1}{a_n-1} - \frac{1}{a_{n+1}-1} \right) = \frac{1}{a_1-1} - \frac{1}{a_{N+1}-1} < \frac{1}{a_1-1} = 1.$$

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