RMO 2018a

Exercise 1 (RMO 2018a P2,). Let n be a natural number. Find all real numbers x satisfying the equation

$$\sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}} = \frac{n(n+1)}{4}.$$

Solution 1. Let x be a real number satisfying the above equation. It follows that $x \neq 0$. Note that

$$\begin{aligned}
\frac{n(n+1)}{4} &= \sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}} \\
&= \sum_{k=1}^{n} \frac{k}{x^k + \frac{1}{x^k}} \\
&\leq \sum_{k=1}^{n} \frac{k}{|x|^k + \frac{1}{|x|^k}} \\
&\leq \sum_{k=1}^{n} \frac{k}{2} \\
&= \frac{n(n+1)}{4}.
\end{aligned}$$

Consequently, the inequalities in the intermediate steps are equalities. This shows that x is positive and $|x| = 1$, which gives $x = 1$. Also note that the given equation holds if $x = 1$. Hence, $x = 1$ is the only real solution of the given equation. ■

Exercise 2 (RMO 2018a P3,). For a rational number r, its *period* is the length of the smallest repeating block in its decimal expansion, for example, the number $r = 0.123123123\ldots$ has period 3. If S denotes the set of all rational numbers of the form $r = 0.\overline{abcdefgh}$ having period 8, find the sum of all elements in S.

Solution 2. The sum of all rational numbers of the form $0.\overline{abcdefgh}$ is equal to

$$\sum_{0\le i_1,i_2,\ldots,i_8\le 9}\left(\frac{i_1}{10}+\frac{i_2}{10^2}+\cdots+\frac{i_8}{10^8}\right)\frac{1}{1-\frac{1}{10^8}}.$$

Note that the rational numbers of the form $0.\overline{abcdefgh}$ which have period less than 8, are of period 1 or 2 or 4, and hence, they are of the form $0.\overline{abcd}$. The sum of such rational numbers is equal to

$$\sum_{0\le i_1,i_2,i_3,i_4\le 9}\left(\frac{i_1}{10}+\frac{i_2}{10^2}+\cdots+\frac{i_4}{10^4}\right)\frac{1}{1-\frac{1}{10^4}}.$$

It follows that the sum of the rational numbers of the given form having period 8 is equal to

$$\begin{aligned}
&\sum_{0\le i_1,i_2,\ldots,i_8\le 9}\left(\frac{i_1}{10}+\frac{i_2}{10^2}+\cdots+\frac{i_8}{10^8}\right)\frac{1}{1-\frac{1}{10^8}}\\
&-\sum_{0\le i_1,i_2,i_3,i_4\le 9}\left(\frac{i_1}{10}+\frac{i_2}{10^2}+\cdots+\frac{i_4}{10^4}\right)\frac{1}{1-\frac{1}{10^4}}\\
&=10^7\left(\sum_{i=0}^{9}i\right)\left(\frac{1}{10}+\frac{1}{10^2}+\cdots+\frac{1}{10^8}\right)\frac{1}{1-\frac{1}{10^8}}\\
&-10^3\left(\sum_{i=0}^{9}i\right)\left(\frac{1}{10}+\frac{1}{10^2}+\cdots+\frac{1}{10^4}\right)\frac{1}{1-\frac{1}{10^4}}\\
&=\frac{10^7}{9}\left(\sum_{i=0}^{9}i\right)-\frac{10^3}{9}\left(\sum_{i=0}^{9}i\right)\\
&=5\times 10^7-5\times 10^3\\
&=(50000-5)\times 10^3\\
&=49995000.
\end{aligned}$$

■

Solution 3. Note that the rational numbers of the form $0.\overline{abcdefgh}$, not having period 8, have period 1 or 2 or 4, and hence, are of the form $0.\overline{abcd}$. This shows that there are precisely 10^8-10^4 rational numbers of the form $0.\overline{abcdefgh}$, having period 8.

Observe that

$$0.\overline{abcdefgh}\mapsto 0.\overline{\mathsf{abcdefgh}},$$

where $\mathsf{x}=9-x$ for any $x\in\{a,b,c,d,e,f,g,h\}$, defines a map of order two from S to S, and this map has no fixed points. This shows that each element

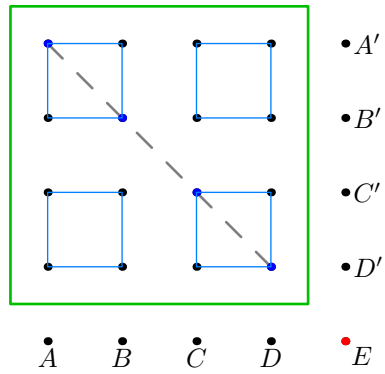


Figure 1: RMO 2018a P4, Exercise 3

of S can be paired with its image under this map, and it produces $\frac{1}{2}|S|$ many pairs. Note that the sum of the elements of any such pair is equal to

$$0.\overline{999999} = 1.$$

Consequently, the sum of the elements of S is equal to

$$\frac{1}{2}|S| = \frac{10^8 - 10^4}{2} = 49995000.$$

■

Exercise 3 (RMO 2018a P4,). Let E denote the set of 25 points (m, n) in the xy-plane, where m, n are natural numbers, $1 \leq m \leq 5, 1 \leq n \leq 5$. Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set E of the form $(a, b), (a + k, b), (a + k, b + k), (a, b + k)$ for some positive integer k such that at least three of these four points have the same colour. (That is, there always exist four points in the set E which form the vertices of a square and having at least three points of the same colour.)

Walkthrough —

(a) Assume that the conclusion is false, and by interchanging the colors (if necessary), assume that there are more red points than the blue ones.

(b) Show that one of the four corners is red, and assume without loss of generality that the point E (as in Fig. 1) is red.

(c) Considering the number of red points in each of the sets $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$, $\{D, D'\}$, prove that these four sets together contain at most 4 red points, and conclude that the green square (as in Fig. 1) contains at least 8 red points.

(d) Consider the blue squares in the green square, to show that the green square contains exactly 8 red points.

(e) Conclude that there are exactly 13 red points, and each of the sets $\{A, A'\}$, $\{B, B'\}$, $\{C, C'\}$, $\{D, D'\}$ contains exactly one red point, and the green square contains exactly 8 red points.

(f) For each point on the dashed diagonal (as in Fig. 1), consider the square having this point and E as the endpoints of one of its diagonals to show that the points on the dashed diagonal are blue.

(g) Note that each point within the green square lying outside the dashed diagonal, is a vertex of a square having two of its remaining vertices on the dashed diagonal. Conclude that each such point is red.

(h) Obtain a contradiction to conclude that the assumption is false.

Solution 4. On the contrary, let us assume that there is no axes-parallel square having at least three vertices of the same color.

Note that at least 13 among those 25 points are of the same color. Without loss of generality, assume that those are red. By our hypothesis, it follows that among the four vertices at the corners, there is at least one red vertex. Without loss of generality[1], let us assume that the bottom-right vertex is red, and denote this vertex by E (as in Fig. 1).

Note that each of the sets $\{A, A'\}, \{B, B'\}, \{C, C'\}, \{D, D'\}$ (as in Fig. 1) contains at most one red point, otherwise, we can form an axes-parallel square with at least three vertices of the same color. Consequently, the green square (as in Fig. 1) contains at least $13 - 5 = 8$ red points.

If the green square contains at least 9 red points, then at least one of the four blue squares (as in Fig. 1) contains at least three red points, which is not the case by our hypothesis. Hence, the green square contains exactly 8 red points. Consequently, there are precisely 13 red points among the 25 points, and each of the sets $\{A, A'\}, \{B, B'\}, \{C, C'\}, \{D, D'\}$ contains exactly one red point.

For any given point on the dashed diagonal (as in Fig. 1), consider the square having this point and E as the endpoints of one of its diagonals. Note that one of the other two vertices of this square is red. By our hypothesis, it follows that all the points on the gray diagonal (as in Fig. 1) are blue.

Note that each point within the green square lying outside the dashed diagonal, is a vertex of a square having two of its remaining vertices on the dashed diagonal. This shows that all the points within the green square, lying outside the gray diagonal, are red. Hence, there exists an axes-parallel square, all whose vertices are red. This contradicts our hypothesis, and hence, it follows that there exists an axes-parallel square having at least three vertices of the same color. ■

[1] Convince yourself that there is **indeed no loss of generality** in assuming so.

Exercise 4 (RMO 2018a P5,). Find all natural numbers n such that $1 + [\sqrt{2n}]$ divides $2n$. (For any real number x, $[x]$ denotes the largest integer not exceeding x.)

Solution 5. Let n be a positive integer such that $1 + [\sqrt{2n}]$ divides $2n$. Write

$$2n = m^2 + k$$

where m is a positive integer, and k is an integer satisfying $0 \leq k \leq 2m$. Note that

$$1 + [\sqrt{2n}] = m + 1$$

holds. Since $1 + [\sqrt{2n}]$ divides $2n$, it follows that

$$\begin{aligned} \frac{2n}{1 + [\sqrt{2n}]} &= \frac{m^2 + k}{m + 1} \\ &= \frac{(m+1)^2 - 2(m+1) + k + 1}{m + 1} \\ &= m - 1 + \frac{k + 1}{m + 1} \end{aligned}$$

is an integer, which shows that $m + 1$ divides $k + 1$. Using $0 \leq k \leq 2m$, we obtain $k = m$, which yields

$$n = \frac{m(m+1)}{2}.$$

Conversely, if n is equal to

$$\frac{m(m+1)}{2}$$

for some positive integer m, then $1 + [\sqrt{2n}]$ divides $2n$.

This proves that the positive integers satisfying the given condition are precisely the integers of the form

$$\frac{m(m+1)}{2}$$

for some positive integer m. ■