RMO 2017b

Maharashtra and Goa

Exercise 1 (RMO 2017b P1)**.** Consider a chessboard of size 8 units $\times$ 8 units (i.e. each small square on the board has a side length of 1 unit). Let S be the set of all the 81 vertices of all the squares on the board. What is the number of line segments whose vertices are in S, and whose length is a positive integer? (The segments need not be parallel to the sides of the board.)

Summary — A segment having vertices in S and length a positive integer, is horizontal or vertical, or the hypotenuse of a right-angled triangle whose smaller sides are parallel to the sides of the board. To count such right-angled triangles, note that they cannot have a too large hypotenuse.

Walkthrough —

(a) Determine the number of the horizontal segments with vertices in S and whose lengths are positive integers.

(b) By symmetry, the number of such vertical segments is equal to the above.

(c) To determine the slanted ones, note that such a slanted segment is the hypotenuse of a right-angled triangle whose smaller sides are parallel to the sides of the board, and have integer lengths. Note that the diagonal of an 8×8 chessboard has length $8\sqrt{2} < 12$. Thus, the only right-angled triangles, that can be fit within the board having sides parallel to the sides of the board and of integer length, have side lengths equal to $(3, 4, 5)$, $(4, 3, 5)$, $(6, 8, 10)$, $(8, 6, 10)$.

(d) Does a symmetry argument help? For instance, flipping around a diagonal, and then flipping around an axis (i.e. a line parallel to one of the sides of the board and dividing the board in two equal halves).

Solution 1. Note that within each horizontal line, there are $8-\ell+1$ horizontal segments of length ℓ for any $1 \leq \ell \leq 8$. This shows that the number of the horizontal segments with vertices in S and whose lengths are positive integers is equal to

$$8 \times (1 + 2 + \cdots + 8) = \frac{1}{2} 8^2 \cdot 9.$$

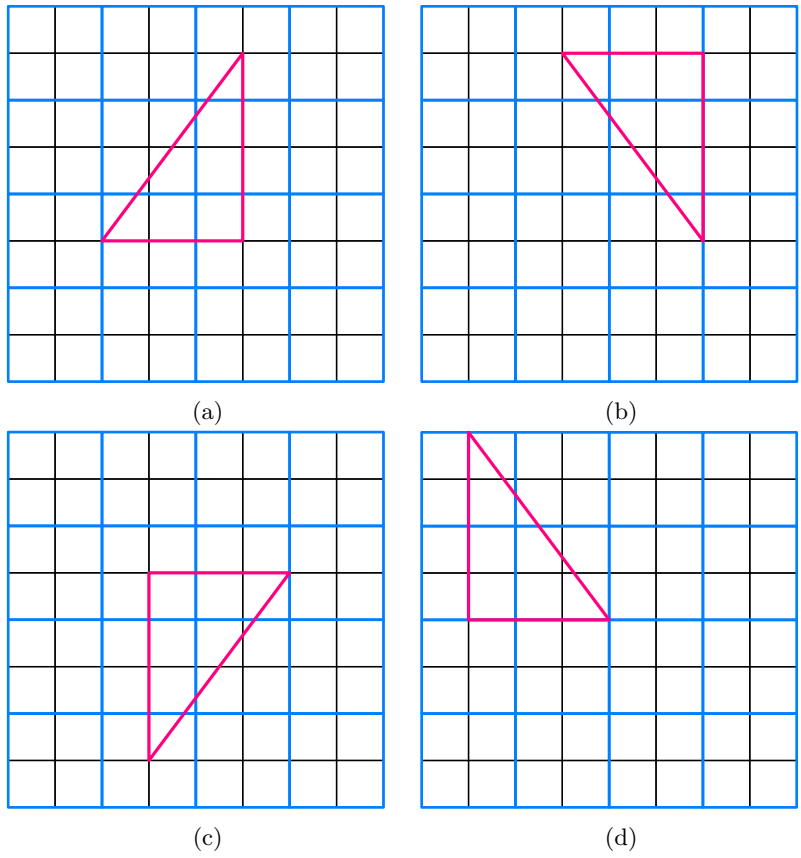


Figure 1: India RMO 2017b P1 — Several configurations of triangles with side lengths $(3, 4, 5)$, Exercise 1

By symmetry, the number of such vertical segments is also equal to $\frac{1}{2}8^2 \cdot 9$. Hence, there are $8^2 \cdot 9$ segments parallel to the sides of the board, which have vertices in S and whose lengths are positive integers.

Note that the diagonal of an 8×8 chessboard has length $8\sqrt{2} < 12$. Thus, the only right-angled triangles, that can be fit within the board having sides parallel to the sides of the board and of integer length, have the side lengths equal to $(3, 4, 5)$, $(4, 3, 5)$, $(6, 8, 10)$, $(8, 6, 10)$. The number of such right-angled triangles, having the side lengths equal to $(3, 4, 5)$, is equal to

$$4 \times (8 - 3 + 1) \times (8 - 4 + 1).$$

The number of such right-angled triangles, having the side lengths equal to the remaining triples, can be expressed in a similar way. This shows that the total

count of such triangles is

$$2 \times 4 \times ((8-3+1) \times (8-4+1) + (8-6+1) \times (8-8+1)) = 360.$$

Hence, the number of the line segments with the stated property is

$$8^2 \cdot 9 + \frac{1}{2} \cdot 360 = 756.$$

■

Remark. Note that the above argument considers the line segments whose endpoints are distinct. There are 81 line segments having equal end-points and end-points lying in S.

Exercise 2 (RMO 2017b P2)**.** For any positive integer n, let $d(n)$ denote the number of positive divisors of n; and let $\varphi(n)$ denote the number of elements from the set $\{1, 2, \ldots, n\}$ that are coprime to n. (For example, $d(12) = 6$ and $\varphi(12) = 4$.) Find the smallest positive integer n such that $d(\varphi(n)) = 2017$.

Solution 2. Let n be the least among the positive integers (if any) satisfying $d(\varphi(n)) = 2017$. Observe that $\varphi(n)$ is not equal to 1. Let $\varphi(n) = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ denote the factorization of $\varphi(n)$ into the product of the powers of distinct primes. We obtain

$$(\alpha_1 + 1) \ldots (\alpha_r + 1) = 2017.$$

Noting that 2017 is a prime, it follows that $r = 1$, and $\alpha_1 = 2016$, that is, $\varphi(n)$ is the 2016-th power of a prime. Since $\varphi(n)$ is larger than 1, it follows that it is an even number, and hence, $\varphi(n)$ is equal to 2^{2016}. Note that if an odd prime q divides n, then q^2 does not divide n, and $q-1$ is a power of 2. It follows that n is equal to $2^k p_1 \ldots p_m$ where k, m are nonnegative integers, and $p_1 < \cdots < p_m$ are distinct odd primes, and $p_i - 1$ is a power of 2 for any $1 \le i \le m$.

Claim — If p is an odd prime such that $p-1$ is a power of 2, then p is equal to $2^{2^i} + 1$ for some integer $i \ge 0$.

Proof of the Claim. It suffices to consider the case $p \ge 5$. Write $p = 2^a + 1$. Note that $a \ge 2$. If a is odd, then p is divisible by 3, and $p > 3$, which is a contradiction to the primality of p. This shows that a is even. If a is not a power of 2, then writing $a = 2^b c$ for some positive integers b, c, with c an odd integer, it follows that p is divisible by $2^{2^b} + 1$, which shows that c is equal to 1. This completes the proof. □

Note that the above Claim shows that for any $1 \le i \le m$, p_i is equal to $2^{2^{a_i}} + 1$ for some nonnegative integer a_i.

Claim — For no integer $1 \le i \le m$, the prime p_i is equal to $2^{2^5}+1$.

Proof of the Claim. Note that $641 = 5 \cdot 2^7 + 1$ is a prime. This gives $5 \cdot 2^7 \equiv -1 \bmod 641$, which yields $5^4 \cdot 2^{28} \equiv 1 \bmod 641$. Using $641 = 2^4 + 5^4$, it follows that $2^{32}+1$ is divisible by 641. Since $2^{32}+1$ is larger than $5 \cdot 2^7 + 1 = 641$, the Claim follows. □

Let us first consider the case that $k = 0$. Since $\varphi(n) = 2^{2016} < 2^{2^{11}}$ holds, it follows that

$$p_m \le 2^{2^{10}} + 1.$$

The first Claim shows that $\{p_1, \dots, p_m\}$ is a subset of

$$\left\{2^{2^i} + 1 \,|\, 0 \le i \le 10\right\}.$$

This shows that

$$\begin{aligned} 2^{2016} &= \varphi(p_1 \dots p_m) \\ &\le \prod_{0 \le i \le 10, i \ne 5} 2^{2^i} \quad \text{(using the second Claim)} \\ &= 2^{1+2+4+8+16+64+128+256+512+1024} \\ &= 2^{2015}, \end{aligned}$$

which is impossible.

Let us consider the case that $k \ge 1$. If $m \ge 1$, then note that

$$n > 2^k (p_1 - 1)(p_2 - 1) \dots (p_m - 1)$$

and

$$\varphi(n) = \varphi(2^k (p_1 - 1)(p_2 - 1) \dots (p_m - 1))$$

hold, which contradicts the minimality of n. This shows that $m = 0$, that is, n is a power of 2, and hence, $n = 2^{2017}$.

Noting that

$$d(\varphi(2^{2017})) = d(2^{2016}) = 2017$$

holds, it follows that the smallest positive integer n satisfying $d(\varphi(n)) = 2017$ is 2^{2017}. ■

Exercise 3 (RMO 2017b P3). Let $P(x)$ and $Q(x)$ be polynomials of degree 6 and degree 3 respectively, such that

$$P(x) > Q(x)^2 + Q(x) + x^2 - 6 \text{ for all } x \in \mathbb{R}.$$

If all the roots of $P(x)$ are real numbers, then prove that there exist two roots of $P(x)$, say α, β such that $|\alpha - \beta| < 1$.

Solution 3. For any real number x, note that

$$P(x) > \left(Q(x) + \frac{1}{2}\right)^2 + x^2 - \frac{25}{4} \geq x^2 - \frac{25}{4}$$

holds. This implies that any root α of $P(x)$ satisfies $\alpha^2 < \frac{25}{4}$. Since $P(x)$ is a polynomial of degree 6 and all its roots are real, it follows that its six roots lie in the interval $(-5/2, 5/2)$, which is the union of the five intervals

$$(-5/2, -3/2), [-3/2, -1/2), [-1/2, 1/2), [1/2, 3/2), [3/2, 5/2).$$

Hence, there are two roots α, β of $P(x)$ such that $|\alpha - \beta| < 1$. ■

Exercise 4 (RMO 2017b P5)**.** If $a, b, c, d \in \mathbb{R}$ such that $a > b > c > d > 0$ and $a + d = b + c$; then prove that

$$\frac{(a+b) - (c+d)}{\sqrt{2}} > \sqrt{a^2 + b^2} - \sqrt{c^2 + d^2}.$$

Solution 4. Put $m = a + d$. Using $a + d = b + c$ and $a > b > c > d > 0$, it follows that for some real numbers $0 < y < x < m$,

$$a = m + x, b = m + y, c = m - y, d = m - x$$

hold. Note that the given inequality reduces to

$$\sqrt{2}(x + y) > \sqrt{(m+x)^2 + (m+y)^2} - \sqrt{(m-x)^2 + (m-y)^2}.$$

Observe that

$$\begin{aligned}
&\sqrt{(m+x)^2 + (m+y)^2} - \sqrt{(m-x)^2 + (m-y)^2} \\
&= \frac{(m+x)^2 + (m+y)^2 - (m-x)^2 - (m-y)^2}{\sqrt{(m+x)^2 + (m+y)^2} + \sqrt{(m-x)^2 + (m-y)^2}} \\
&= \frac{4m(x+y)}{\sqrt{(m+x)^2 + (m+y)^2} + \sqrt{(m-x)^2 + (m-y)^2}}.
\end{aligned}$$

Hence, it suffices to prove that

$$\sqrt{(m+x)^2 + (m+y)^2} + \sqrt{(m-x)^2 + (m-y)^2} > 2\sqrt{2}m.$$

Note that

$$\begin{aligned}
&\sqrt{(m+x)^2 + (m+y)^2} + \sqrt{(m-x)^2 + (m-y)^2} \\
&> \frac{m+x+m+y}{\sqrt{2}} + \frac{m-x+m-y}{\sqrt{2}} \\
&= 2\sqrt{2}m
\end{aligned}$$

hold. This completes the proof. ■