RMO 2017a

Exercise 1 (RMO 2017a P2,). Show that the equation

$$a^3 + (a+1)^3 + \cdots + (a+6)^3 = b^4 + (b+1)^4$$

has no solutions in integers a, b.

> **Walkthrough** — Read it modulo a suitable integer ★. Are there any intuitions about the choice of ★?

Solution 1. Note that the fourth powers of the integers $0, 1, 2, 3, 4, 5, 6$ are congruent to $0, 1, 2, 4, 4, 2, 1$ modulo 7 respectively. The shows that the sum of the fourth powers of two consecutive integers is congruent to one of

$$0+1, 1+2, 2+4, 4+2, 2+1, 1+0$$

modulo 7. Hence, the sum of the fourth powers of two consecutive integers is not divisible by 7. Also note the cubes of seven consecutive integers is congruent to

$$0^3 + 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3$$

modulo 7, which is congruent to

$$1^3 + 2^3 + 3^3 + (-3)^3 + (-2) + (-1)^3 = 0$$

modulo 7. This shows that the sum of the cubes of seven consecutive integers is not equal to the sum of the fourth powers of two consecutive integers. This completes the proof. ■

Exercise 2 (RMO 2017a P3,). Let $P(x) = x^2 + \frac{x}{2} + b$ and $Q(x) = x^2 + cx + d$ be two polynomials with real coefficients such that $P(x)Q(x) = Q(P(x))$ for all real x. Find all real roots of $P(Q(x)) = 0$.

Solution 2. Let α be a root of $P(x)$ in $\mathbb{C}$. Substituting $x = \alpha$ in $P(x)Q(x) = Q(P(x))$, we obtain $Q(0) = 0$, which shows that $d = 0$. This gives

$$P(x)(x^2 + cx) = P(x)(P(x) + c),$$

which yields $P(x) + c = x^2 + cx$. It follows that $c = \frac{1}{2}, b = -\frac{1}{2}$.

Note that the roots of $P(x)$ are $-1, \frac{1}{2}$. Hence, any root β of $P(Q(x)) = 0$ satisfies $Q(\beta) = -1$ or $Q(\beta) = \frac{1}{2}$. Since the discrimint of $Q(x) + 1$ is negative, it follows that if β is a real root of $P(Q(x)) = 0$, then $Q(\beta) = \frac{1}{2}$, which gives $\beta = -1$ or $\beta = 1/2$.

Note that $P(\frac{1}{2}) = 0$ and $-1, \frac{1}{2}$ are the roots of $Q(x) = 1/2$. This shows that the real roots of $P(Q(x)) = 0$ are $-1, \frac{1}{2}$. ■

Exercise 3 (RMO 2017a P4,). Consider n^2 unit squares in the xy-plane centred at the point (i, j) with integer coordinates for $1 \le i \le n, 1 \le j \le n$. It is required to colour each unit square in such a way that whenever $1 \le i < j \le n$ and $1 \le k < \ell \le n$, the three squares with centres at (i, k), (j, k), (j, ℓ) have distinct colours. What is the least possible number of colours needed?

Remark. For such problems, it is often useful to first work out a special case.

Walkthrough —

(a) First, work out a simple case in order to gain insight for the general case.

(b) One may consider the squares below a suitable diagonal.

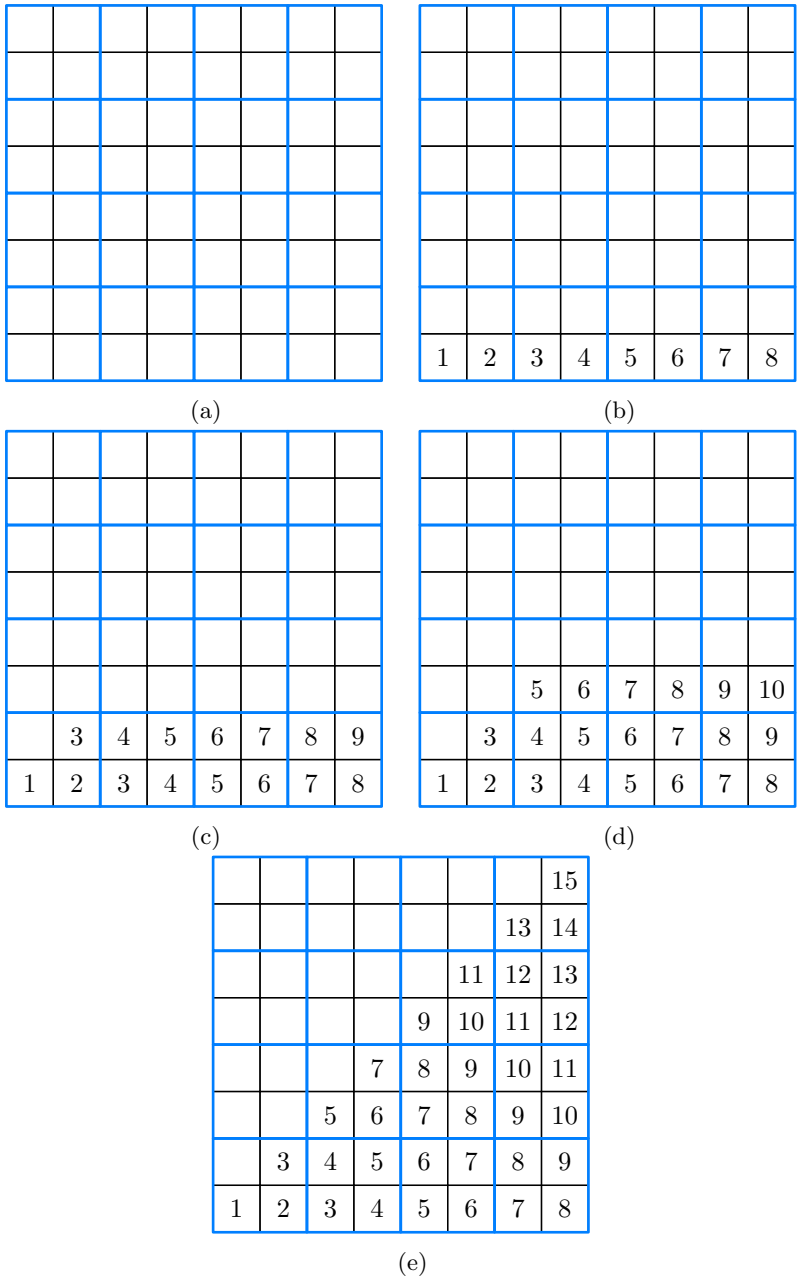
(c) Extend to the general case!

First, let's work on it. Let us begin with the case $n = 8$. First, let us try to color the unit squares with as few colors as we can. This may provide some insight for the least number of colors required (for the case $n = 8$ and also possibly for the general case).

Note that any two unit squares at the bottom row of Fig. 1a have pairwise distinct colors. Let's apply the colors $1, \ldots, 8$ to these squares as in Fig. 1b. Note that in the second last row, all the unit squares except the first one, have colors different from those of the unit squares of the bottom row. Moreover, these seven unit squares have distinct colors. Let's apply the colors $3, \ldots, 9$ to these squares as in Fig. 1c. Similarly, in the third last row, the $8 - 3 + 1$ unit squares lying to the right, have colors different from those of the unit squares have been colored so far. Moreover, these $8 - 3 + 1$ unit squares have distinct colors. Let's apply another set of $8 - 3 + 1$ new colors (for example, $5, \ldots, 10$) to these squares as in Fig. 1d. One may continue this process to yield the coloring as in Fig. 1e.

In the remaining square in the second last row, we may use a color which has been used in that row, for instance, the color 2, as in Fig. 2a. In the third

The content posted here and at this blog by Evan Chen are quite useful.



(a) (b) (c) (d) (e)

Figure 1: India RMO 2017a P4, Exercise 3

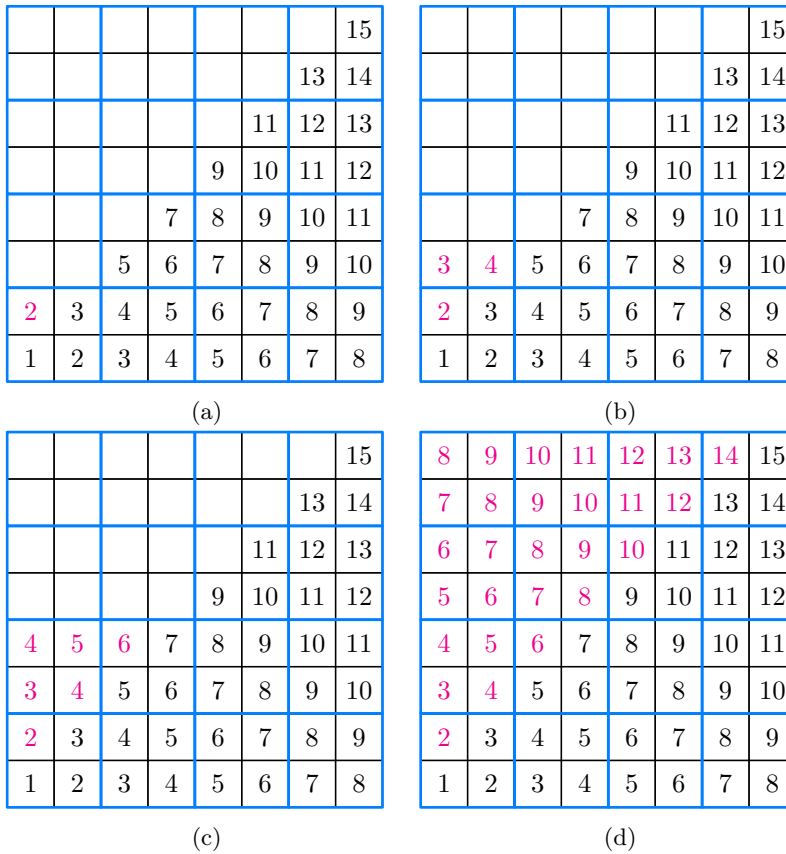


Figure 2: India RMO 2017a P4, Exercise 3

last row, the colors $3, 4$ may be used as in Fig. 2b. In the fourth last row, the colors smaller than 7 may be used as in Fig. 2c. One may continue this process to yield the coloring as in Fig. 2d. Note that the coloring as in Fig. 2d **does satisfy** the required condition. ♣

> **Remark.** The **preceding argument alone** does not guarantee that the least possible number of colors have been used in Fig. 2d.

Skip this part for the first reading —

> **Remark.** The **preceding argument alone** does not guarantee that the least possible number of colors have been used in Fig. 2d. It could have been the case that in our very first attempt of working out a special case, we might have proceeded in way which would have used more colors than what is required. For instance, consider the following approach.
>
> In fact, that is what I (the author) did in a prior attempt, and then I could not prove that the number of colors that one may require to color the squares satisfying the required condition is at least as large as the number of colors used in the approach below. This led to think again, which yield the argument above.

¶ A prior attempt that used more colors than the minimum number of colors required. Let us begin with the case $n = 8$. First, let us try to color the unit squares with as few colors as we can. This may provide some insight for the least number of colors required (for the case $n = 8$ and also possibly for the general case).

Note that any two unit squares at the bottom row of Fig. 3a have pairwise distinct colors. Let us apply the colors $1, \ldots, 8$ to these squares as in Fig. 3b. Note that in the second last row, all the unit squares except the first one, have colors different from those of the unit squares of the bottom row. Moreover, these seven unit squares have distinct colors. Let us apply the colors $9, \ldots, 15$ to these squares as in Fig. 3c. Similarly, in the third last row, the $8 - 3 + 1$ unit squares lying to the right, have colors different from those of the unit squares have been colored so far. Moreover, these $8 - 3 + 1$ unit squares have distinct colors. Let us apply another set of $8 - 3 + 1$ new colors (for example, $16, \ldots, 21$) to these squares as in Fig. 3d. One may continue this process to yield the coloring as in Fig. 3e.

In the remaining square in the second last row, we may use a color which has been used in that row, for instance, the color 8, as in Fig. 4a. In the third last row, the colors $14, 15$ may be used as in Fig. 4b. In the fourth last row, the colors smaller than 22 may be used as in Fig. 4c. One may continue this process to yield the coloring as in Fig. 4d. Note that the coloring as in Fig. 4d does satisfy the required condition. ♣

Now that we have gained some experience, we may proceed to the general case as follows.

Bogus Solution. Suppose we have colored the n^2 unit squares using r colors so that the required condition is met. Consider the diagonal joining the upper right corner with the lower left corner. Note that the unit squares of the bottom row have distinct colors. Moreover, the unit squares lying to the right of the above-mentioned diagonal and contained in a row except the bottom one, have colors different from the unit squares contained in the rows below and lying to the right of that diagonal. Hence, we require at least

$$n + (n-1) + (n-2) + \cdots + 2 + 1 = \frac{1}{2}n(n+1)$$

colors.

Moreover, similar to the coloring as in Fig. 4, we may color the n^2 unit squares using $\frac{1}{2}n(n+1)$ colors so that required condition is satisfied.

This proves that the least number of colors required is $\frac{1}{2}n(n+1)$.

Question. Where does the above go wrong?

Now that we have gained some idea, we may proceed to the general case as follows.

Solution 3. If $n = 1$, then using one color works. Let us assume[1] that $n \geq 2$.

Suppose we have colored the n^2 unit squares using r colors so that the required condition is met. Consider the diagonal joining the upper right corner with the lower left corner. Note that the unit squares of the bottom row have distinct colors. Moreover, the unit squares lying on the rightmost column except the bottom square, have colors different from those of the squares lying on the bottom row. Hence, we require at least

$$n + (n-1) = 2n - 1$$

colors.

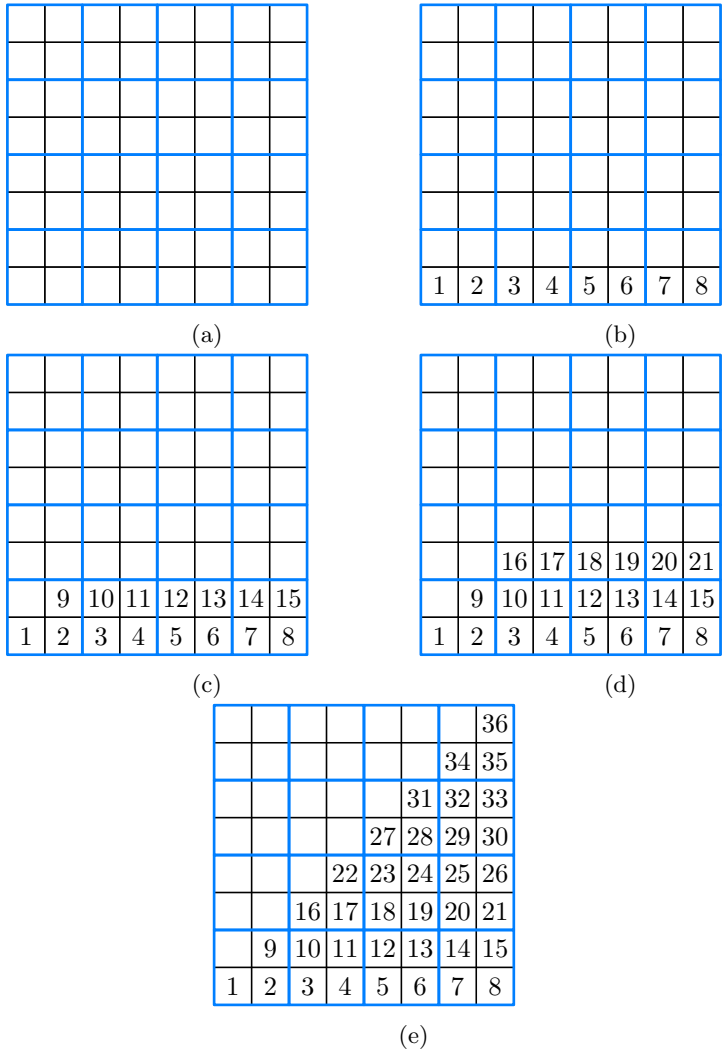
Next, we show that there is a coloring of the n^2 unit squares using $2n-1$ colors satisfying the required condition. Indeed, if the colors $i, i+1, \ldots, i+n-1$ are applied to the squares lying in the i-th last row in an increasing order from the left to the right for all $1 \leq i \leq n$, then the squares in any row have distinct colors and so are the squares in any column, and hence, it is a coloring satisfying the required condition.

This shows that the least number of colors required is $2n-1$. ■

Exercise 4 (RMO 2017a P6,). Let x, y, z be real numbers, each greater than 1. Prove that

$$\frac{x+1}{y+1} + \frac{y+1}{z+1} + \frac{z+1}{x+1} \leq \frac{x-1}{y-1} + \frac{y-1}{z-1} + \frac{z-1}{x-1}$$

[1] The reason for considering the case $n = 1$ separately will become clear from the rest of the argument. Is it clear what would go wrong with the rest of the argument if $n = 1$?



(a) (b) (c) (d) (e)

Figure 3: India RMO 2017a P4, Exercise 3

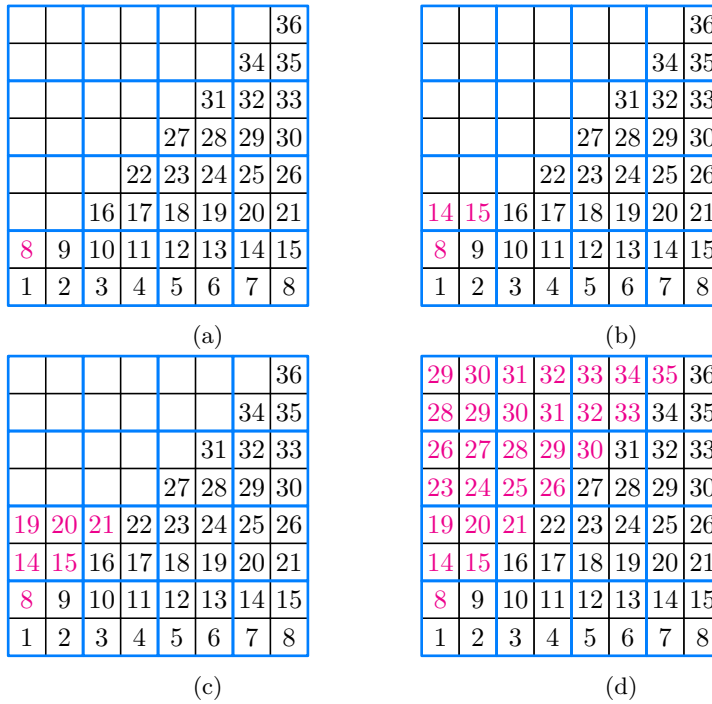


Figure 4: India RMO 2017a P4, Exercise 3

Solution 4. Note that both sides of the above inequality remain invariant under cyclic permutations of x, y, z. Therefore, without loss of generality, we assume that $x \geq y$ and $x \geq z$. Note that

$$\begin{aligned}\frac{x-1}{y-1} - \frac{x+1}{y+1} &= \frac{(x-1)(y+1)-(x+1)(y-1)}{y^2-1} \\ &= \frac{2(x-y)}{y^2-1}.\end{aligned}$$

Similarly, it follows that

$$\begin{aligned}\frac{y-1}{z-1} - \frac{y+1}{z+1} &= \frac{2(y-z)}{z^2-1}, \\ \frac{z-1}{x-1} - \frac{z+1}{x+1} &= \frac{2(z-x)}{x^2-1}.\end{aligned}$$

It suffices to prove that

$$\frac{x-y}{y^2-1} + \frac{y-z}{z^2-1} + \frac{z-x}{x^2-1} \geq 0,$$

which is equivalent to

$$\frac{x}{y^2-1} + \frac{y}{z^2-1} + \frac{z}{x^2-1} \geq \frac{x}{x^2-1} + \frac{y}{y^2-1} + \frac{z}{z^2-1},$$

which follows from the rearrangement inequality since x, y, z and $\frac{1}{x^2-1}, \frac{1}{y^2-1}, \frac{1}{z^2-1}$ are sorted in the opposite order. ■