RMO 2016g

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Exercise 1 (RMO 2016g P1,). Suppose in a given collection of 2016 integers, the sum of any 1008 integers is positive. Show that sum of all 2016 integers is positive.

> **Walkthrough** — Think about it!

Exercise 2 (RMO 2016g P2,). On a stormy night ten guests came to dinner party and left their shoes outside the room in order to keep the carpet clean. After the dinner there was a blackout, and the guests leaving one by one, put on at random, any pair of shoes big enough for their feet. (Each pair of shoes stays together). Any guest who could not find a pair big enough spent the night there. What is the largest number of guests who might have had to spend the night there?

> **Walkthrough** — What happens if a person having shoe of the smallest size wears a shoe of the largest size, and next, if a person having shoe of the second smallest size wears a shoe of the second largest size, and it continues?

Solution 1. If a person having shoe of the k-th smallest size wears a shoe of the k-th largest size for $k = 1, 2, 3, 4, 5$, and if the sizes of the shoes of the ten persons are distinct, then none of the remaining five persons will not find a big enough shoe.

We claim that if k persons have left for some $1 \leq k \leq 4$, then one of the remaining $10 - k$ persons will find a big enough shoe. Let us denote the sizes of the shoes of the guests by $s_1, s_2, \ldots, s_{10}$. If the claim is false, then for some $1 \leq k \leq 4$, there are $10 - k$ guests whose shoes have sizes larger than some $10 - k$ numbers among $s_1, \ldots, s_{10}$. So, some $10 - k$ numbers among $s_1, \ldots, s_{10}$ are larger than some $10 - k$ numbers among $s_1, \ldots, s_{10}$. It follows that

$$10 - k + 10 - k \leq 10,$$

which shows that $k \geq 5$. This contradicts the bound $k \leq 4$. This proves the claim. Consequently, at most five guests may need to spend the night there.

We conclude that the largest number of the guests who might need to spend the night there is five. ■

Exercise 3 (RMO 2016g P3,). a, b, c, d are integers such that $ad + bc$ divides each of a, b, c and d. Prove that $ad + bc = \pm 1$.

Remark. What happens when a, b, c, d are equal to zero?

Solution 2. Note that there exist integers p, q, r, s such that

$$\begin{aligned} a &= p(ad + bc), \\ b &= q(ad + bc), \\ c &= r(ad + bc), \\ d &= s(ad + bc) \end{aligned}$$

holds. This yields

$$ad + bc = (ps + qr)(ad + bc)^2.$$

If $ad + bc = 0$, then all of the integers a, b, c, d are equal to zero, which is impossible. This shows that $ad + bc$ is nonzero, and hence, we obtain

$$(ps + qr)(ad + bc) = 1,$$

which implies

$$ad + bc = \pm 1.$$

■

Exercise 4 (RMO 2016g P6,). Positive integers a, b, c satisfy $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. Prove that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{41}{42}$. Also prove that equality in fact holds in the second inequality.

Solution 3. Let a, b, c be positive integers satisfying

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1. \tag{1}$$

Since $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ is symmetric with respect to a, b, c, by reordering them (if necessary), we may (and do) assume that $a \leq b \leq c$. Using Eq. (1), it follows that $a \geq 2$.

Let us consider the case that $a = 3$. Using Eq. (1) it follows that one of b, c is greater than 3, and hence

$$\frac{1}{3} + \frac{1}{3} + \frac{1}{4} = \frac{2}{3} + \frac{1}{4}$$

The content posted here and at this blog by Evan Chen are quite useful.

$$\begin{aligned} &= \frac{9}{12} \\ &= \frac{3}{4} \\ &< \frac{41}{42} \end{aligned}$$

holds.

Now, let us consider the case that $a = 2$. Using Eq. (1), it follows that the integers b, c are greater than or equal to 3. If $b = 3$, then using Eq. (1), it follows that $c > 6$, which shows that

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{1}{2} + \frac{1}{3} + \frac{1}{c} \\ &\leq \frac{5}{6} + \frac{1}{7} \\ &= \frac{41}{42} \end{aligned}$$

holds. If $b = 4$, then using Eq. (1), it follows that $c > 4$, which shows that

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{c} \\ &\leq \frac{3}{4} + \frac{1}{5} \\ &= \frac{19}{20} \\ &\leq \frac{41}{42} \end{aligned}$$

holds. If $b \geq 5$ holds, then using Eq. (1), it follows that

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &\leq \frac{1}{2} + \frac{2}{b} \\ &\leq \frac{1}{2} + \frac{2}{5} \\ &= \frac{9}{10} \\ &\leq \frac{41}{42} \end{aligned}$$

holds. This proves the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{41}{42}.$$

Observing that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$$

holds, it follows that equality holds in the above inequality for some choice of a, b, c. ■

Exercise 5 (RMO 2016g P8,). At some integer points a polynomial with integer coefficients take values $1, 2$ and 3. Prove that there exist not more than one integer at which the polynomial is equal to 5.

Solution 4. Denote the polynomial by $P(x)$. On the contrary, let us assume that there are at least two distinct integers where $P(x)$ takes the value 5.

Let a, b, c be integers such that

$$P(a) = 1, P(b) = 2, P(c) = 3.$$

Note that $a - b$ divides $P(a) - P(b)$, $b - c$ divides $P(b) - P(c)$. It follows that $a - b = \pm 1, b - c = \pm 1$. Since a, b are of opposite parity, and so are the integers b, c, we obtain that a, c are of the same parity. Noting that $c - a$ divides $P(c) - P(a) = 2$, it follows that $c - a = \pm 2$. Combining this with $a - b = \pm 1, b - c = \pm 1$, we get $a - b = b - c = 1$ or $a - b = b - c = -1$.

This shows that $P(b-1) = 1, P(b) = 2, P(b+1) = 3$ holds or $P(b+1) = 1, P(b) = 2, P(b-1) = 3$ holds. Note that in the first case, the polynomial $R(x) := P(x - b)$ takes the values $1, 2, 3$ at the integers $-1, 0, 1$ respectively. In the second case, the polynomial $S(x) = P(-x + b)$ takes the values $1, 2, 3$ at the integers $-1, 0, 1$ respectively. This proves that there is a polynomial $Q(x)$ with integer coefficients which takes the values $1, 2, 3$ at $-1, 0, 1$ respectively.

From the hypothesis, it follows that there are distinct integers i, j such that $Q(i) = Q(j) = 5$. Note that $i - 1$ divides $Q(i) - Q(1) = 2$, i divides $Q(i) - Q(0) = 3$, $i + 1$ divides $Q(i) - Q(-1) = 4$. Since i divdes 3, we obtain $i = \pm 1, \pm 3$. Using $Q(-1) = 1, Q(1) = 3$, we get $i \neq -1, i \neq 1$. This gives $i = \pm 3$. Noting that $i - 1$ divides 2, we obtain $i \neq -3$, and hence $i = 3$. Similarly, it follows that $j = 3$. ■