# RMO 2016f

## Maharashtra and Goa

**Exercise 1** (RMO 2016f P1, ). Find distinct positive integers $n_1 < n_2 < \cdots < n_7$ with the least possible sum, such that their product $n_1 \times n_2 \times \cdots \times n_7$ is divisible by 2016.

**Solution 1.** Note that

$$2016 = 32 \times 63 = 2^5 \cdot 3^2 \cdot 7$$

holds.

Let $n_1 < n_2 < \cdots < n_7$ be distinct positive integers satisfying the given conditions. Note that $n_i \geq i$ holds for all $1 \leq i \leq 7$, which gives

$$n_1 + n_2 + \cdots + n_7 \geq 1 + 2 + 3 + 4 + 5 + 6 + 7.$$

Note that the product of the distinct integers $1, 2, 3, 4, 6, 7, 8$ is equal to $2^5 \cdot 3^2 \cdot 7$. This implies that

$$n_1 + n_2 + \cdots + n_7 \leq 3 + 1 + 2 + 3 + 4 + 5 + 6 + 7. \tag{1}$$

Note that $n_4 > 4$ implies that

$$n_1 + n_2 + \cdots + n_7 \geq 1 + 2 + 3 + 5 + 6 + 7 + 8 = 4 + 1 + 2 + 3 + 4 + 5 + 6 + 7,$$

which is impossible. This gives that $n_4 \leq 4$, and consequently,

$$n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4$$

holds.

If

$$n_1 + n_2 + \cdots + n_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7$$

holds, then $n_i$ is equal to $i$ for any $1 \leq i \leq 7$, and this is impossible since 7! is not divisible by the divisor $2^5$ of 2016. This proves that

$$n_1 + n_2 + \cdots + n_7 > 1 + 2 + 3 + 4 + 5 + 6 + 7,$$

which implies that $n_i > i$ for some $1 \leq i \leq 7$, and hence $n_i > i$ for some $i \geq 5$. This shows that

$$n_1 + n_2 + \cdots + n_7 \geq 1 + 2 + 3 + 4 + 6 + 7 + 8 = 3 + 1 + 2 + 3 + 4 + 5 + 6 + 7.$$

Using Eq. (1), we obtain

$$n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4, n_5 = 6, n_6 = 7, n_7 = 8.$$

■

**Exercise 2** (RMO 2016f P2, ). At an international event there are 100 countries participating, each with its own flag. There are 10 distinct flagpoles at the stadium, labelled $\#1, \#2, \ldots, \#10$ in a row. In how many ways can all the 100 flags be hoisted on these 10 flagpoles, such that for each $i$ from 1 to 10, the flagpole $\#i$ has at least $i$ flags? (Note that the vertical order of the flagpoles on each flag is important)

**Solution 2.** Let us determine the number of solutions of

$$x_1 + \cdots + x_{10} = 100$$

in the positive integers satisfying $x_i \geq i$ for all $1 \leq i \leq 10$. Note that the solutions are in one-to-one correspondence with the solutions of

$$y_1 + \cdots + y_{10} = 100 - (1 + 2 + \cdots + 9)$$

in the positive integers. Hence, there are $\binom{54}{9}$ solutions of

$$x_1 + \cdots + x_{10} = 100$$

in the positive integers satisfying $x_i \geq i$ for all $1 \leq i \leq 10$. Note that corresponding to each such solution there are 100! arrangements of the flags. Hence the flags can be hoisted in $100! \times \binom{54}{9}$ ways. ■

**Exercise 3** (RMO 2016f P3, ). Find all integers $k$ such that all roots of the following polynomial are also integers:

$$f(x) = x^3 - (k-3)x^2 - 11x + (4k-8).$$

**Summary** — Try to somehow **eliminate** $k$ (for instance, by evaluating $f$ at a suitable integer), and come up with a factorization of an (**nonzero**) integer in terms of the roots of $f$. This would imply that there are only a finitely many possibilites for the roots of $f$, provided $k$ is an integer with the stated properties. Under some **non-degeneracy condition**[a], we would obtain some nontrivial algebraic condition on $k$. Next, the precise choices for $k$ satisfying this condition can be plugged into $f$ to find out for which one of them, the



The content posted here and at this blog by Evan Chen are quite useful.

polynomial $f$ possesses integer roots only.

[a]It just means that "**if life is nice!**"

**Solution 3.** Let $k$ be an integer such that all roots of the polynomial

$$x^3 - (k-3)x^2 - 11x + (4k-8)$$

are integers. Denote the roots of this polynomial by $\alpha, \beta, \gamma$. It follows that

$$\alpha + \beta + \gamma = k - 3, \quad \alpha\beta + \beta\gamma + \gamma\alpha = -11, \quad \alpha\beta\gamma = -(4k-8).$$

Since $\alpha\beta + \beta\gamma + \gamma\alpha = -11$, it follows that at most one of $\alpha, \beta, \gamma$ is even. Since the product $\alpha\beta\gamma$ is divisible by 4, it follows that exactly one of $\alpha, \beta, \gamma$ is even, and it is a multiple of 4. Reordering $\alpha, \beta, \gamma$ if necessary, we assume that 4 divides $\alpha$. Using $\alpha\beta + \beta\gamma + \gamma\alpha = -11$, it follows that $\beta\gamma \equiv 1 \pmod 4$.

Note that

$$f(2) = 8 - 4k + 12 - 22 + 4k - 8 = -10.$$

This gives $(2-\alpha)(2-\beta)(2-\gamma) = -10$. Since

$$\alpha \equiv 0 \pmod 4, \quad \beta \equiv 1 \pmod 4,$$

holds, it follows that

$$2 - \alpha \equiv 2 \pmod 4, \quad (2-\beta)(2-\gamma) \equiv 1 \pmod 4.$$

This combined with $(2-\alpha)(2-\beta)(2-\gamma) = -10$ shows that $2-\alpha$ is equal to one of

$$-10, -2, 2, 10,$$

and it is not equal to any of $2, 10$. This gives that $\alpha = 12$ or $\alpha = 4$. Since $\alpha$ is a root of $f(x)$ and $\alpha^2 \neq 4$, we obtain

$$k = \frac{\alpha^3 + 3\alpha^2 - 11\alpha - 8}{\alpha^2 - 4} = \alpha + 3 + \frac{4 - 7\alpha}{\alpha^2 - 4}.$$

Note that $\alpha \neq 12$, otherwise, $k$ would not be an integer. For $\alpha = 4$, we obtain $k = 7 - 2 = 5$.

Finally, note that for $k = 5$,

$$f(x) = x^3 - 2x^2 - 11x + 12,$$

which factorizes into $(x+3)(x-1)(x-4)$, and hence has integer roots.

We conclude that $k = 5$ is the only integer such that the required condition holds. ■

Here is another approach to Exercise 3, which seems to be more general (or, perhaps, *somewhat* more general in sense that it gives an **algorithmic approach** which would work in several other cases. This does not require finding constraints stemming from congruence conditions. Note that coming up with constraints originating from congruence conditions requires to first have an **appropriate modulus** which would provide any presumably useful constraints, without knowing a priori whether any such **modulus** exists.).

**Summary** — Express $k$ in terms of an integer root of such an equation. Show that such an integer root cannot be **too large**. Use it to narrow down the number of possibilities for $k$ (recall that in absolute values, a higher degree polynomial dominates a lower degree polynomial at arguments which are large enough in absolute values). Check for which of these values the given polynomial has integer roots only.

**Solution 4.** Let $k$ be an integer such that all the roots of the polynomial

$$f(x) = x^3 - (k-3)x^2 - 11x + (4k-8)$$

are integers[1]. Let $\alpha$ denote an integer root of $f(x)$. Note that $\alpha^2 \neq 4$, and hence,

$$k = \frac{\alpha^3 + 3\alpha^2 - 11\alpha - 8}{\alpha^2 - 4} = \alpha + 3 + \frac{4 - 7\alpha}{\alpha^2 - 4}.$$

This implies that

$$\left|\alpha^2 - 4\right| \leq |4 - 7\alpha|,$$

or equivalently,

$$\begin{cases} -(7\alpha - 4) \leq \alpha^2 - 4 \leq 7\alpha - 4 & \text{if } \alpha \geq \frac{4}{7}, \\ 7\alpha - 4 \leq \alpha^2 - 4 \leq 4 - 7\alpha & \text{if } \alpha \leq \frac{4}{7}. \end{cases}$$

This shows that $\alpha$ lies in

$$([1, 7] \cup [-8, 0]) \setminus \{-2, 2\}.$$

Since $k$ is an integer, it follows that $\alpha^2 - 4$ divides $7\alpha - 4$, and consequently, $\alpha$ is equal to one of

$$-3, 1, 4, 7.$$

Substituting these values of $\alpha$ in

$$k = \alpha + 3 + \frac{4 - 7\alpha}{\alpha^2 - 4},$$

it follows that $k = 5$ or $k = 9$. Note that for $k = 9$, we have

$$f(x) = x^3 - 6x^2 - 11x + 28 = (x - 7)(x^2 + x - 4),$$

and not all its roots are integers. It follows that $k = 5$.

Finally, note that for $k = 5$,

$$f(x) = x^3 - 2x^2 - 11x + 12,$$

which factorizes into $(x + 3)(x - 1)(x - 4)$, and hence has integer roots.

We conclude that $k = 5$ is the only integer such that the required condition holds. ■

[1] For several of the steps of the argument, it suffices to assume **only** that $f(x)$ has at least one integer root.

**Exercise 4** (RMO 2016f P5, ). Let $x, y, z$ be non-negative real numbers such that $xyz = 1$. Prove that

$$(x^3 + 2y)(y^3 + 2z)(z^3 + 2x) \geq 27.$$

**Solution 5.** Applying the AM-GM inequality, we obtain

$$\begin{aligned}
&(x^3 + 2y)(y^3 + 2z)(z^3 + 2x) \\
&\geq (3\sqrt[3]{x^3y^2})(3\sqrt[3]{y^3z^2})(3\sqrt[3]{z^3x^2}) \\
&= 27.
\end{aligned}$$

■