RMO 2016e

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Exercise 1 (RMO 2016e P2,). Consider a sequence $(a_k)_{k\geq 1}$ of natural numbers defined as follows: $a_1 = a$ and $a_2 = b$ with $a, b > 1$ and $\gcd(a, b) = 1$ and for all $k > 0$, $a_{k+2} = a_{k+1} + a_k$. Prove that for all natural numbers n and k, $\gcd(a_n, a_{n+k}) < \frac{a_k}{2}$.

> **Remark.** Note that for the above inequality, it is necessary that $a_1 \geq 3$. In fact, **if $a_1 = 2$, then the above inequality is not strict for $k = 1$.**

Solution 1. Consider the sequence $\{f_n\}_{n\geq 1}$ satisfying the recurrence relation

$$f_{n+2} = f_{n+1} + f_n, \quad \text{for any } n \geq 1,$$

and $f_1 = 0, f_2 = 1$. Note that

$$a_n = af_{n-1} + bf_n, \quad \text{for any } n \geq 2.$$

> **Claim —** For any integers $n \geq 2, k \geq 1$,
>
> $$f_{n+k-1} = f_{n-1}f_k + f_nf_{k+1}$$
>
> holds.

Proof of the Claim. It suffices to consider the case $k \geq 2$. Note that

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix}$$

holds for $n = 1$, and if it holds for $n = m$ for some integer $m \geq 1$, then it holds for $n = m + 1$ since

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{m+1} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f_m & f_{m+1} \\ f_{m+1} & f_{m+2} \end{pmatrix}$$

$$= \begin{pmatrix} f_{m+1} & f_{m+2} \\ f_{m+1}+f_m & f_{m+2}+f_{m+1} \end{pmatrix}$$
$$= \begin{pmatrix} f_{m+1} & f_{m+2} \\ f_{m+2} & f_{m+3} \end{pmatrix}$$

holds. This shows that the above holds for any positive integer n. Consequently, for any integers $n, k \geq 2$, it follows that

$$\begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix} \begin{pmatrix} f_k & f_{k+1} \\ f_{k+1} & f_{k+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{n-1+k} = \begin{pmatrix} f_{n+k-1} & f_{n+k} \\ f_{n+k} & f_{n+k+1} \end{pmatrix},$$

which yields

$$f_{n+k-1} = f_{n-1}f_k + f_n f_{k+1}.$$

□

Let n, k be positive integers. Using the above Claim, we obtain

$$\begin{aligned} a_{n+k} &= af_{n+k-1} + bf_{n+k} \\ &= a(f_{n-1}f_k + f_n f_{k+1}) + b(f_{n-1}f_{k+1} + f_n f_{k+2}) \\ &= a(f_{n-1}f_k + f_n f_{k+1}) + b(f_{n-1}f_{k+1} + f_n f_k + f_n f_{k+1}) \\ &= (af_{n-1} + bf_n)f_k + (af_n + bf_{n-1} + bf_n)f_{k+1} \\ &= a_n f_k + (af_n + bf_{n+1})f_{k+1} \\ &= a_n f_k + a_{n+1}f_{k+1}. \end{aligned}$$

This yields

$$\begin{aligned} \gcd(a_n, a_{n+k}) &= \gcd(a_n, a_n f_k + a_{n+1}f_{k+1}) \\ &= \gcd(a_n, a_{n+1}f_{k+1}). \end{aligned}$$

Note that

$$\gcd(a_n, a_{n+1}) = 1$$

holds for $n = 1$, and if it holds for $n = m$ for some positive integer m, then

$$\begin{aligned} \gcd(a_{m+1}, a_{m+2}) &= \gcd(a_{m+1}, a_{m+1} + a_m) \\ &= \gcd(a_{m+1}, a_m) \\ &= 1, \end{aligned}$$

which shows that the integers a_n, a_{n+1} are relatively prime for any positive integer n. If $a_1 \geq 3$, then it follows that

$$\gcd(a_n, a_{n+1}) = 1 < \frac{a_1}{2},$$

that is, for $k = 1$, the inequality

$$\gcd(a_n, a_{n+k}) < \frac{a_k}{2}$$

holds. Moreover, if $k \geq 2$, then using the above, we obtain

$$\begin{aligned}\gcd(a_n, a_{n+k}) &= \gcd(a_n, f_{k+1}) \\ &\leq f_{k+1} \\ &= f_{k-1} + f_k \\ &< \frac{af_{k-1} + bf_k}{2} \\ &= \frac{a_k}{2}.\end{aligned}$$

This completes the proof. ■

Exercise 2 (RMO 2016e P4,). Let a, b, c be positive real numbers such that $a + b + c = 3$. Determine, with certainty, the largest possible value of the expression

$$\frac{a}{a^3 + b^2 + c} + \frac{b}{b^3 + c^2 + a} + \frac{c}{c^3 + a^2 + b}.$$

Solution 2. Note that

$$\begin{aligned}&\frac{a}{a^3 + b^2 + c} + \frac{b}{b^3 + c^2 + a} + \frac{c}{c^3 + a^2 + b} \\ &= \frac{1}{a^2 + \frac{b^2}{a} + \frac{c}{a}} + \frac{1}{b^2 + \frac{c^2}{b} + \frac{a}{b}} + \frac{1}{c^2 + \frac{a^2}{c} + \frac{b}{c}} \\ &\leq \frac{1 + a + ca}{(a + b + c)^2} + \frac{1 + b + ab}{(a + b + c)^2} + \frac{1 + c + bc}{(a + b + c)^2} \\ &\qquad \text{(by the Cauchy–Schwarz inequality)} \\ &= \frac{3 + a + b + c + ab + bc + ca}{(a + b + c)^2} \\ &\leq \frac{6 + \frac{(a+b+c)^2}{3}}{9} \\ &= 1.\end{aligned}$$

Also note that if $a = b = c = 1$, then $a + b + c = 3$ and

$$\frac{a}{a^3 + b^2 + c} + \frac{b}{b^3 + c^2 + a} + \frac{c}{c^3 + a^2 + b} = 1.$$

This shows that the largest possible value of the given expression is equal to 1. ■

Exercise 3 (RMO 2016e P5,).

(i) A 7-tuple $(a_1, a_2, a_3, a_4, b_1, b_2, b_3)$ of pairwise distinct positive integers with no common factor is called a *shy* tuple if

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = b_1^2 + b_2^2 + b_3^2$$

and for all $1 \leq i < j \leq 4$ and $1 \leq k \leq 3$, $a_i^2 + a_j^2 \neq b_k^2$. Prove that there exist infinitely many shy tuples.

(ii) Show that 2016 can be written as a sum of squares of four distinct natural numbers.

Solution 3. For any integer $n \geq 2$,

$$(3, 4, 3n, 4n, 3(n^2-1), 6n, 4(n^2+1))$$

is a 7-tuple of pairwise distinct positive integers, and these integers have no common factor. This proves part (i).

Note that $2016 = 2^5 \cdot 3^2 \cdot 7$ holds. Observe that $2 \cdot 3^2 \cdot 7$ can be expressed as $4^2 + 5^2 + 6^2 + 7^2$. This gives

$$2016 = 2^5 \cdot 3^2 \cdot 7 = 16^2 + 20^2 + 24^2 + 28^2,$$

expressing 2016 as the sum of the squares of four distinct natural numbers. ■

Remark. The identity

$$(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$$

can also be used to look for shy tuples. Note that

$$\begin{aligned}(x-3y)^2 + (x-y)^2 + (x+y)^2 + (x+3y)^2 &= 2(x^2+9y^2) + 2(x^2+y^2)\\ &= (2x)^2 + 20y^2\\ &= (2x)^2 + (2y)^2 + (4y)^2.\end{aligned}$$

This leads to considering the tuple $(x-3y, x-y, x+y, x-3y, 2x, 2y, 4y)$. To make it a shy tuple, one can take x, y to be positive integers satisfying $x - 3y \geq 1$, and one can impose the condition that x, y are coprime, along with the additional condition that x, y are of different parity.

Alternatively, one can make use of the following identity.

$$(a+b+c)^2 + (-a+b+c)^2 + (a-b+c)^2 + (a+b-c)^2 = (2a)^2 + (2b)^2 + (2c)^2.$$

For more details, we refer to this AoPS thread.

Exercise 4 (RMO 2016e P6,). A deck of 52 cards is given. There are four suites each having cards numbered $1, 2, \ldots, 13$. The audience chooses some five cards with distinct numbers written on them. The assistant of the magician comes by, looks at the five cards and turns exactly one of them face down and arranges all five cards in some order. Then the magician enters and with an agreement made beforehand with the assistant, he has to determine the face down card (both suite and number). Explain how the trick can be completed.

Solution 4. The following could be an agreement between the magician and the assistant, so that the magician can determine the face down card.

1. Given a collection of five cards, having distinct numbers written on them, the assistant puts them in a row, as follows.

2. Note that among the five cards, at least two are of the same suite.

3. The assistant chooses two cards of the same color, and puts them next to each other, and then keeps the remaining three cards to their right.

4. Note that if the integers $1, 2, \ldots, 13$ are written along the circumference of a circle Γ (in some order that is known to both the magician and the assistant, for instance, in the clockwise direction), then for any two distinct integers i, j lying between 1 and 13, one of them (call it Q) can be reached from the other one (call it P) by traversing from P along the circumference in the clockwise or in the anti-clockwise direction within at most 6 steps.

5. Denote the chosen cards of the same color, by P and Q, so that Q can be reached from P, by traversing along the circumference, in the clockwise or in the anti-clockwise direction, within at most 6 steps. The assistant puts the cards P and Q in the order Q, P (resp. P, Q) if Q can be reached from P, by traversing along the circumference, in the anti-clockwise (resp. clockwise) direction, in $s \leq 6$ steps. The assistant turns the card Q face down.

6. The assistant keeps the remaining three cards to the right of these two cards, in an order, which helps to determine the integer s as follows.

7. Denote the integers written on the remaining three cards by a, b, c with $a < b < c$. The $3! = 6$ permutations of $1, 2, 3$ can be identified with the integers $1, 2, 3, 4, 5, 6$ in some order (known to both the magician and the assistant), and the $3! = 6$ permutations of the three remaining cards can be identified with the permutations of $1, 2, 3$ in an obvious way that is known to both the magician and the assistant (for instance, sending a permutation σ of $\{a, b, c\}$ to the permutation $f \circ \sigma \circ f^{-1}$ of $\{1, 2, 3\}$ where $f : \{a, b, c\} \to \{1, 2, 3\}$ denotes the map sending a, b, c to $1, 2, 3$ respectively.). Under the composition of these two identifications, a positive integer ≤ 6 corresponds to a unique permutation of the three remaining cards. The assistant keeps the three remaining cards to the right of the first two cards, as per the permutation which corresponds to the integer s.

Since the suite of the face down card is same as the face up card which is the first in the row of the five cards, the magician determines the suite of the face down card. The magician knows that the number written on the face down

card can be reached, by traversing along Γ from the number written on the first face up card, in the clockwise (resp. anti-clockwise) direction if the face down card in the second (resp. first) one in the row of five. ■

The content posted here and at this blog by Evan Chen are quite useful.