RMO 2016d

CRMO 4 (Odisha)

Exercise 1 (RMO 2016d P2,). Let a, b, c be positive real numbers such that

$$\frac{ab}{1+bc} + \frac{bc}{1+ca} + \frac{ca}{1+ab} = 1.$$

Prove that

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \geq 6\sqrt{2}.$$

Solution 1. Applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} 1 &= \frac{ab}{1+bc} + \frac{bc}{1+ca} + \frac{ca}{1+ab} \\ &\geq \frac{(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})^2}{3 + ab + bc + ca}, \end{aligned}$$

which yields

$$3 \geq 2\sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c}).$$

Using the AM-GM inequality, we get

$$2\sqrt{abc}\sqrt[3]{\sqrt{abc}} \leq 1,$$

which gives $abc \leq \frac{1}{2\sqrt{2}}$. Note that

$$\begin{aligned} \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} &\geq \frac{3}{abc} \\ &\geq 6\sqrt{2}. \end{aligned}$$

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We refer to **??** and **??**.

Exercise 2 (RMO 2016d P3,). The present ages in years of two brothers A and B, and their father C are three distinct positive integers a, b, and c respectively. Suppose $\frac{b-1}{a-1}$ and $\frac{b+1}{a+1}$ are two consecutive integers, and $\frac{c-1}{b-1}$ and $\frac{c+1}{b+1}$ are two consecutive integers. If $a + b + c \leq 150$ determine a, b and c.

Solution 2. Note that

$$\frac{b-1}{a-1}-\frac{b+1}{a+1}=\frac{(a+1)(b-1)-(a-1)(b+1)}{a^2-1}=\frac{2(b-a)}{a^2-1}$$

holds, and

$$\frac{c-1}{b-1}-\frac{c+1}{b+1}=\frac{(b+1)(c-1)-(b-1)(c+1)}{b^2-1}=\frac{2(c-b)}{b^2-1}$$

holds. Note that if $a>b$, then

$$\frac{2(a-b)}{a^2-1}=1$$

holds, which yields

$$2-2b=(a-1)^2,$$

which gives $a=2$, and $b=0$, which is impossible since $b\geq 1$. This shows that $a\leq b$. It follows that

$$\frac{2(b-a)}{a^2-1}=1,\quad \frac{2(c-b)}{b^2-1}=1$$

hold. This yields

$$2(b-a)=a^2-1,$$
$$2(c-b)=b^2-1,$$

which implies

$$2b=a^2+2a-1,$$
$$2c=b^2+2b-1.$$

Note that a,b are odd, and

$$2(b+1)=(a+1)^2,$$
$$2(c+1)=(b+1)^2$$

holds. This shows that

$$a+b+c+3=a+1+\frac{(a+1)^2}{2}+\frac{(a+1)^4}{8}.$$

Using the bound $a+b+c\leq 150$, we obtain

$$a+1+\frac{(a+1)^2}{2}+\frac{(a+1)^4}{8}\leq 153,$$

which shows that $a\leq 4$. Since a is odd, it follows that $a=3$. It follows that

$$b=7,\quad c=31.$$

The content posted here and at this blog by Evan Chen are quite useful.

Note that the triple $(a, b, c) = (3, 7, 31)$ satisfies $a + b + c \leq 150$, and $\frac{b-1}{a-1} = 3$ and $\frac{b+1}{a+1} = 2$ are two consecutive integers, and $\frac{c-1}{b-1} = 5$ and $\frac{c+1}{b+1} = 4$ are two consecutive integers.

This proves that

$$a = 3, \quad b = 7, \quad c = 31.$$

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Exercise 3 (RMO 2016d P4,). A box contains 4032 answer scripts out of which exactly half have odd number of marks. We choose 2 scripts randomly and, if the scores on both of them are odd number, we add one mark to one of them, put the script back in the box and keep the other script outside. If both scripts have even scores, we put back one of the scripts and keep the other outside. If there is one script with even score and the other with odd score, we put back the script with the odd score and keep the other script outside. After following this procedure a number of times, there are 3 scripts left among which there is at least one script each with odd and even scores. Find, with proof, the number of scripts with odd scores among the three left.

Walkthrough — What happens to the number of scripts with odd scores?

Solution 3. Note that under this process, the number of scripts with odd scores remains unchanged or decreases by two. Consequently, if this process is repeated few times, then the number of scripts with odd scores would remain unchanged or decrease by an even number. Note that the box contains $4032 \times \frac{1}{2} = 2016$ scripts with odd scores. So after following the process certain number of times, the number of scripts with odd number of marks will always be an even number. The number of scripts with odd scores among the three left being at least one, is equal two. ■

Exercise 4 (RMO 2016d P6,).

(i) Prove that if an infinite sequence of strictly increasing positive integers in arithmetic progression has one cube then it has infinitely many cubes.

(ii) Find, with justification, an infinite sequence of strictly increasing positive integers in arithmetic progression which does not have any cube.

Solution 4. Let us first establish part (i). Suppose we are given an infinite sequence of strictly increasing positive integers in arithmetic progression. Let a denote a term of this progression, and assume that $a = b^3$ for some integer b. Let d denote the common difference of the progression. Let k be a positive integer. Write $x = kd$. Note that

$$x^3 + 3x^2b + 3xb^2 \geq \frac{1}{2}x^3 + 3x^2b + \frac{1}{2}x^3 + 3xb^2$$

$$\begin{aligned}
&\geq \frac{1}{2}x^3 + 3x^2b + \frac{1}{2}x^3 + 3xb^2 \\
&\geq \frac{1}{2}x^2(x + 6b) + \frac{1}{2}x(x^2 + b^2) \\
&> 0
\end{aligned}$$

holds if $x > 6\,|b|$. For any integer $k > 6\,|b|$, the integer $x^3 + 3x^2b + 3xb^2$ is a multiple of d, and the term $x^3 + 3x^2b + 3xb^2 + a$ of the given progression is equal to the perfect cube $(x + b)^3$. Consequently, the given progression has infinitely many cubes.

To prove part (ii), note that no even cube is congruent to 2 modulo 8. Hence, the arithmetic progression, having 2 as its first term, and with common difference 8, does not have any cube. ■