RMO 2016c

CRMO 3 (Karnataka and West Bengal)

Exercise 1 (RMO 2016c P2,). Let a, b, c be three distinct positive real numbers such that $abc = 1$. Prove that

$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \geq 3.$$

Solution 1. Note that

$$\begin{aligned}
&\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \\
&= -\frac{a^3(b-c) + b^3(c-a) + c^3(a-b)}{(a-b)(b-c)(c-a)} \\
&= -\frac{a^3(b-c) + b^3(c-a) - c^3(b-c) - c^3(c-a)}{(a-b)(b-c)(c-a)} \\
&= -\frac{(b^3-c^3)(c-a) + (a^3-c^3)(b-c)}{(a-b)(b-c)(c-a)} \\
&= -\frac{b^2+bc+c^2-c^2-a^2-ca}{a-b} \\
&= -\frac{b^2+bc-a^2-ca}{a-b} \\
&= a+b+c \\
&\geq 3\sqrt[3]{abc} \\
&= 3.
\end{aligned}$$

■

Remark. The above argument leads to following somewhat simpler solution. Observing that

$$\begin{aligned}
\frac{1}{(c-a)(c-b)} &= \frac{(c-a)-(c-b)}{(b-a)(c-a)(c-b)} \\
&= \frac{1}{(b-a)(c-b)} - \frac{1}{(b-a)(c-a)},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \\
&= \frac{a^3-c^3}{(a-b)(a-c)} + \frac{b^3-c^3}{(b-c)(b-a)} \\
&= \frac{a^2+ac+c^2}{a-b} - \frac{b^2+bc+c^2}{a-b} \\
&= a+b+c \\
&\geq 3.
\end{aligned}$$

Exercise 2 (RMO 2016c P3,). Let a, b, c, d, e, f be positive integers such that

$$\frac{a}{b} < \frac{c}{d} < \frac{e}{f}.$$

Suppose $af - be = -1$. Show that $d \geq b + f$.

Solution 2. The given conditions yield

$$ad < bc, \quad cf < de.$$

This gives

$$adf \leq bcf - f \leq bde - b - f,$$

which implies

$$d(be - af) \geq b + f.$$

It follows that $d \geq b + f$. ■

Exercise 3 (RMO 2016c P4,). There are 100 countries participating in an olympiad. Suppose n is a positive integer such that each of the 100 countries is willing to communicate in exactly n languages. If each set of 20 countries can communicate in at least one common language, and no language is common to all 100 countries, what is the minimum possible value of n?

Walkthrough —

(a) Choose a country C, and let L_1 denote a language in which C communicates. Note that L_1 is not common to some country C_1. Hence, some language L_2 is common to the countries C, C_1.

(b) The language L_2 is not common to some country C_2. Hence, some language L_3 is common to the countries C, C_1, C_2.

(c) **Continuing** this line of argument, show that there are pairwise distinct languages $L_1, \ldots, L_{20}$ and C communicates in them.

(d) Conclude that $n \geq 20$.

The content posted here and at this blog by Evan Chen are quite useful.

(e) Show that 20 is the minimum possible value of n.

Solution 3.

Claim — The inequality $n \geq 20$ holds.

Proof of the Claim. Let C denote a country. Let L_1 denote a language in which C communicates. By hypothesis, the language L_1 is not common to some country C_1. Some language L_2 is common to the countries C, C_1. By hypothesis, L_2 is not common to some country C_2. Note that some language L_3 is common to the countries C, C_1, C_2.

Suppose for some integer $2 \leq i$, there are countries $C_1, C_2, \ldots, C_i$ and there are languages $L_1, L_2, \ldots, L_{i+1}$ such that for any $1 \leq k \leq i$, the language L_{k+1} is common to the $k+1$ countries $C, C_1, \ldots, C_k$, and L_k is not common to C_k. By the hypothesis, L_{i+1} is not common to some country C_{i+1}. If $i+2 \leq 20$, then the $i+2$ countries $C, C_1, \ldots, C_{i+1}$ have some common language L_{i+2}. By induction, there are countries $C_1, C_2, \ldots, C_{19}$, and there are languages $L_1, \ldots, L_{20}$ such that for any $1 \leq k \leq 19$, the language L_{k+1} is common to the $k+1$ countries $C, C_1, \ldots, C_k$, and that the language L_k is not common to the country C_k. For any $1 \leq k \leq 19$, the language L_{k+1} is common to the $k+1$ countries $C, C_1, \ldots, C_k$, and for any $1 \leq i \leq k$, the language L_i is not common to C_i, and hence, L_{k+1} is not equal to any of $L_1, \ldots, L_k$. This proves that the languages $L_1, \ldots, L_{20}$ are pairwise distinct, and C communicates in them. The Claim follows. □

Claim — One possible value of n is 20.

Proof of the Claim. Let $\mathscr{L}_1, \ldots, \mathscr{L}_{21}$ be pairwise languages. Let $\mathscr{C}_1, \ldots, \mathscr{C}_{100}$ be a choice of 100 countries. Suppose for $1 \leq i \leq 21$, the country $\mathscr{C}_i$ communicates in all the languages $\mathscr{L}_1, \ldots, \mathscr{L}_{21}$ other than $\mathscr{L}_i$. Assume further that for any $22 \leq i \leq 100$, the country $\mathscr{C}_i$ communicates in $\mathscr{L}_1, \ldots, \mathscr{L}_{20}$. Note that each country communicates in exactly 20 languages, and that no language is common to all the 100 countries. Moreover, each set of 20 countries among $\mathscr{C}_1, \ldots, \mathscr{C}_{21}$ share exactly one common language.

Note that at most 19 countries from $\mathscr{C}_1, \ldots, \mathscr{C}_{21}$, share at least two languages among $\mathscr{L}_1, \ldots, \mathscr{L}_{21}$ in common, and hence share at least one language among $\mathscr{L}_1, \ldots, \mathscr{L}_{20}$ in common. Hence, each set of 20 countries among $\mathscr{C}_1, \ldots, \mathscr{C}_{100}$, having at most 19 countries from $\mathscr{C}_1, \ldots, \mathscr{C}_{20}$, share at least one language in common. This proves the Claim. □

Combining the above Claims, it follows that the minimum possible value of n is 20. ■

Exercise 4 (RMO 2016c P6,).

(a) Given any natural number N, prove that there exists a strictly increasing sequence of N positive integers in harmonic progression.

(b) Prove that there cannot exist a strictly increasing infinite sequence of positive integers which is in harmonic progression.

Walkthrough — If a, c are positive integers, then note that

$$a(a+c), 2ac, c(a+c)$$

form a harmonic progression. Moreover, if a, c, e are three positive integers forming a harmonic progression, then it follows that there exist positive integers λ, μ, b, d such that $\lambda a, b, \lambda c$, and $\mu c, d, \mu e$ are harmonic progressions. Consequently,

$$\lambda\mu a, \mu b, \lambda\mu c, \lambda d, \lambda\mu e$$

is a harmonic progression.

Solution 4. To prove part (a), let us establish the following claim.

Claim — If a, c are positive integers, then the integers

$$a(a+c), 2ac, c(a+c)$$

form a harmonic progression. Consequently, given two positive integers, they can be scaled by a positive integers, such that the products form the first and the last term of a 3-term harmonic progression.

Proof of the Claim. The claim follows since

$$\frac{1}{a(a+c)} + \frac{1}{c(a+c)} = \frac{2}{2ac}$$

holds. □

Claim — Let $k \geq 3$ be a positive integer. For any positive integers $x_1, x_3, x_5, \ldots, x_{2k-1}$ in a harmonic progression, there exist positive integers $x_2, x_4, x_6, \ldots, x_{2k-2}$ and a positive integer λ such that the integers

$$\lambda x_1, x_2, \lambda x_3, x_4, \lambda x_5, x_6, \ldots, x_{2k-2}, \lambda x_{2k-1}$$

form a harmonic progression. Moreover, if $x_1, x_3, \ldots, x_{2k-1}$ is strictly increasing, then so is the above sequence.

Proof of the Claim. Let $x_1, x_3, x_5, \ldots, x_{2k-1}$ be positive integers in a harmonic progression. For any $2 \leq i \leq k$, there exist positive integers λ_i and x_{2i-2}

such that the integers $\lambda_i x_{2i-3}, x_{2i-2}, \lambda_i x_{2i-1}$ are in a harmonic progression. Write $\lambda = \prod_{2 \leq i \leq k} \lambda_i$. It follows that for any $2 \leq i \leq k$, the integers $\lambda x_{2i-3}, \lambda x_{2i-2}/\lambda_i, \lambda x_{2i-1}$ are in a harmonic progression. Consequently, the integers

$$\lambda x_1, \lambda x_2/\lambda_2, \lambda x_3, \lambda x_4/\lambda_3, \lambda x_5, \lambda x_6/\lambda_4, \lambda x_7, \ldots, \lambda x_{2k-2}/\lambda_k, \lambda x_{2k-1}$$

form a harmonic progression. This proves the first part of the Claim. The second part follows. □

Since $3, 4, 6$ is a harmonic progression of strictly increasing positive integers, by the above Claim, it follows that there exist harmonic progressions of arbitrarily large length, formed by stricly increasing sequences of positive integers. This proves part (a).

To prove part (b), let us assume on the contrary that there exists a strictly increasing infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers, which is in a harmonic progression. For any integer $k \geq 2$, note that a_1, a_k, a_{2k-1} is in harmonic progression, and it follows that

$$\begin{aligned} a_k &= 2\frac{a_1 a_{2k+1}}{a_1 + a_{2k+1}} \\ &= 2a_1 \left(1 - \frac{a_1}{a_1 + a_{2k+1}}\right) \\ &< 2a_1. \end{aligned}$$

This shows that the terms of the sequence $a_1, a_2, a_3, \ldots$ are bounded from the above. This is impossible since $a_1, a_2, a_3, \ldots$ is a strictly increasing sequence of positive integers. This completes the proof. ■