RMO 2016b

CRMO 2 (Telangana)

Exercise 1 (RMO 2016b P2, , RMO 2016a P2)**.** Let a, b, c be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$

Prove that $abc \leq \frac{1}{8}$.

Solution 1. Put

$$x = \frac{a}{1+a}, y = \frac{b}{1+b}, z = \frac{c}{1+c}.$$

It follows that x, y, z are positive real numbers satisfying $x + y + z = 1$. Note that

$$a = \frac{x}{1-x}, b = \frac{y}{1-y}, c = \frac{z}{1-z}.$$

We need to shows that $8abc \leq 1$, which is equivalent to $8xyz \leq (1-x)(1-y)(1-z)$. Using $x+y+z = 1$, it follows that the above inequality is equivalent to

$$8xyz \leq (x+y)(y+z)(z+x),$$

which follows from the AM-GM inequality. ■

> **Remark.** A careful reading of the above argument shows that the substitution
>
> $$\frac{x}{x+y+z} = \frac{a}{1+a}, \frac{y}{x+y+z} = \frac{b}{1+b}, \frac{z}{x+y+z} = \frac{c}{1+c},$$
>
> or equivalently, the substitution
>
> $$a = \frac{x}{y+z}, b = \frac{y}{z+x}, c = \frac{z}{x+y}$$
>
> reduces $8abc \leq 1$ to the inequality $8xyz \leq (x+y)(y+z)(z+x)$.

We refer to **??**.

Exercise 2 (RMO 2016b P3,). For any natural number n, expressed in base 10, let $S(n)$ denote the sum of all digits of n. Find all natural numbers n such that $n^3 = 8S(n)^3 + 6nS(n) + 1$.

Solution 2. Let n be a natural number satisfying the given condition. Using $S(n) \equiv n \bmod 9$, we obtain

$$7n^3 + 6n^2 + 1 \equiv 0 \bmod 9.$$

Note that

$$\begin{aligned} 7n^3 + 6n^2 + 1 &= (n+1)(7n^2 - n + 1) \\ &\equiv (n+1)(n+1)(7n+1) \bmod 9 \\ &\equiv 7(n+1)(n+1)(n+4) \bmod 9 \end{aligned}$$

holds, which implies that 3 divides $n+1$ or 9 divides $n+4$.

Note that

$$8S(n)^3 + 6nS(n) + 1 - n^3 = (2S(n))^3 + (-n)^3 + 1^3 - 3(2S(n))(-n)$$

holds. This shows that

$$2S(n) + 1 = n,$$

or

$$2S(n) = 1 = -n$$

holds, and hence, we get

$$2S(n) + 1 = n.$$

Suppose n has d digits. Note that

$$\begin{aligned} 10^{d-1} &\le n \\ &= 2S(n) + 1 \\ &\le 2 \cdot 9d + 1 \end{aligned}$$

holds. Observe that $10^{3-1} > 18 \cdot 3 + 1$ holds, and if k is a positive integer satisfying $10^{k-1} > 18k + 1$, then

$$\begin{aligned} \frac{10^k}{18k+19} &> \frac{10(18k+1)}{18k+19} \\ &> 1 \end{aligned}$$

holds. By induction, it follows that $d \le 2$, that is, n has at most two digits. It follows that $S(n) \le 18$.

Note that $n = 2S(n) + 1$ implies that n is odd. Moreover, 3 divides $n+1$ or 9 divides $n+4$. Combining these with the congruence $S(n) \equiv n \bmod 9$, we obtain that $S(n)$ is equal to one of $5, 11, 17$. Note that none of the integers $5, 11$ satisfies $n = 2S(n) + 1$. This shows that $n = 17$. Since $n = 17$ satisfies $2S(n) + 1 = n$, it also satisfies the given condition. It follows that $n = 17$ is the only natural number satisfying the given condition. ■

The content posted here and at this blog by Evan Chen are quite useful.

Exercise 3 (RMO 2016b P4,). How many 6-digit natural numbers containing only the digits $1, 2, 3$ are there in which 3 occurs exactly twice and the number is divisible by 9?

Solution 3. Note that

$$\begin{aligned}
1+1+1+1+3+3 &\equiv 1 \pmod 9,\\
1+1+1+2+3+3 &\equiv 2 \pmod 9,\\
1+1+2+2+3+3 &\equiv 3 \pmod 9,\\
1+2+2+2+3+3 &\equiv 4 \pmod 9,\\
2+2+2+2+3+3 &\equiv 5 \pmod 9
\end{aligned}$$

hold. This shows that 9 does not divide the sum of $a, b, c, d, 3, 3$ for any choice of $a, b, c, d \in \{1, 2\}$. Hence, there are no 6-digit natural numbers satisfying the given conditions. ■

Exercise 4 (RMO 2016b P6,). Show that the infinite arithmetic progression $\langle 1, 4, 7, 10, \ldots\rangle$ has infinitely many 3-term subsequences in harmonic progression such that for any two such triples $\langle a_1, a_2, a_3\rangle$ and $\langle b_1, b_2, b_3\rangle$ in harmonic progression, one has

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}.$$

Walkthrough —

(a) Try to find out a few 3-term harmonic progressions within the given sequence.

(b) Note that $1, 4$ cannot be extended to such a progression, since

$$\frac{1}{c} = \frac{2}{4} - \frac{1}{1}$$

has no solution in the positive integers.

(c) Note that $4, 7$ extends to the harmonic progression $4, 7, 28$ (solve the analogue of the above equation.).

(d) How to obtain more of such progressions? Let's have a look at $4, 7, 28$, and try to find **"why it works"**. Note that

$$\frac{2}{7} - \frac{1}{4} = \frac{2\cdot 4 - 7}{4\cdot 7} = \frac{1}{4\cdot 7}.$$

This suggests to consider $n, 2n-1, n(2n-1)$, and check that it **does** form a harmonic progression.

(e) Does there exist enough positive integers n such that the terms of the harmonic progression $n, 2n-1, n(2n-1)$ are congruent to 1 modulo 3?

Solution 4. Note that if n is a positive integer and $n \equiv 1 \pmod 3$, then the integers $n, 2n-1, n(2n-1)$ are congruent to 1 modulo 3, and they form a harmonic progression. Making n vary over the positive integers of the form $3k+1$, we obtain infinitely many 3-term harmonic progressions. Moreover, two such triples satisfy the desired inequality. ■