RMO 2016a

CRMO 1 (Mumbai)

Exercise 1 (RMO 2016a P2, , RMO 2016b P2)**.** Let a, b, c be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$

Prove that $abc \leq \frac{1}{8}$.

Solution 1. Put

$$x = \frac{a}{1+a}, y = \frac{b}{1+b}, z = \frac{c}{1+c}.$$

It follows that x, y, z are positive real numbers satisfying $x + y + z = 1$. Note that

$$a = \frac{x}{1-x}, b = \frac{y}{1-y}, c = \frac{z}{1-z}.$$

We need to shows that $8abc \leq 1$, which is equivalent to $8xyz \leq (1-x)(1-y)(1-z)$. Using $x+y+z = 1$, it follows that the above inequality is equivalent to

$$8xyz \leq (x+y)(y+z)(z+x),$$

which follows from the AM-GM inequality. ■

> **Remark.** A careful reading of the above argument shows that the substitution
>
> $$\frac{x}{x+y+z} = \frac{a}{1+a}, \frac{y}{x+y+z} = \frac{b}{1+b}, \frac{z}{x+y+z} = \frac{c}{1+c},$$
>
> or equivalently, the substitution
>
> $$a = \frac{x}{y+z}, b = \frac{y}{z+x}, c = \frac{z}{x+y}$$
>
> reduces $8abc \leq 1$ to the inequality $8xyz \leq (x+y)(y+z)(z+x)$.

We refer to **??**.

Exercise 2 (RMO 2016a P3,). For any natural number n, expressed in base 10, let $S(n)$ denote the sum of all digits of n. Find all natural numbers n such that $n = 2S(n)^2$.

Solution 2. Let n be a natural number satisfying the given conditions. Using $n \equiv S(n) \bmod 9$, it follows that

$$n \equiv 2n^2 \bmod 9,$$

which gives

$$2n^2 \equiv n \bmod 3,$$

which implies that n is congruent to one of $0, 2$ modulo 3. Consequently, n is congruent to one of $0, 3, 6, 2, 5, 8$ modulo 9. Using $n \equiv 2n^2 \bmod 9$, we obtain that n is congruent to one of $0, 5$ modulo 9.

Note that if n has d digits, then

$$10^{d-1} \leq n = 2S(n)^2 \leq 2(9d)^2$$

holds. Note that $10^{5-1} > 2(9 \cdot 5)^2$ holds, and if k is a positive integer such that $10^{k-1} > 2(9k)^2$ holds, then we obtain

$$\begin{aligned} \frac{10^k}{2(9(k+1))^2} &> \frac{10 \times 2(9k)^2}{2(9(k+1))^2} \\ &= \frac{10}{(1+1/k)^2} \\ &\geq \frac{10}{2^2} \\ &> 1. \end{aligned}$$

It follows that $10^{k-1} > 2(9k)^2$ holds for any integer $k \geq 5$. We obtain that $d \leq 4$, that is, n has at most four digits. This gives that $S(n) \leq 36$. Since $S(n) \equiv n \bmod 9$ and n is congruent to one of $0, 5$ modulo 9, it suffices to consider the following cases.

Let us consider the case that $S(n)$ is a multiple of 9. Then $S(n)$ is equal to one of $9, 18, 27, 36$, and n is equal to one of $162, 648, 1458, 2592$. Note that none of the integers $1458, 2592$ satisfies the given condition. This shows that n is one of $162, 648$.

Let us now consider the case that $S(n)$ is equal to one of $5, 14, 23, 32$. Then $S(n)$ is equal to one of $50, 392, 1058, 2048$. Note that none of $1058, 2048$ satisfies the given condition, which implies that n is equal to one of $50, 392$.

It follows that n is equal to one of the integers

$$50, 162, 392, 648.$$

Note that the above integers satisfy the given condition. Hence, the above are precisely all the natural numbers satisfying $n = S(n)^2$. ■

The content posted here and at this blog by Evan Chen are quite useful.

Exercise 3 (RMO 2016a P4,). Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits.

Solution 3. The required number is equal to

$$\begin{aligned}&\binom{6}{3} \times 5^3 \times 5^3 - \binom{5}{3} \times 5^3 \times 5^2\\ &= 20 \times 5^6 - 10 \times 5^5\\ &= 5^6(20 - 2)\\ &= 281250.\end{aligned}$$

■

Exercise 4 (RMO 2016a P6,). Let $\langle a_1, a_2, a_3, \ldots \rangle$ be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression.

> **Walkthrough** — For any positive integer d, note that
>
> $$1, 1 + d, (1 + d)^2, (1 + d)^3, \ldots$$
>
> is a geometric progression, and it is a subsequence of the arithmetic progression with first term equal to 1, and having common difference d. Does this help?

Solution 4. For any positive integers a and d, note that

$$a, a(1 + d), a(1 + d)^2, \ldots$$

forms a geometric progression, and it is a subsequence of the arithmetic progression, with first term equal to a and common difference d. ■