RMO 2015f

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Exercise 1 (RMO 2015f P1, Putnam 1998 B1)**.** Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x \in \mathbb{R}^+$.

Solution 1. Note that

$$\begin{aligned}\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)} &= \frac{(x+1/x)^6-(x^3+1/x^3)^2}{(x+1/x)^3+(x^3+1/x^3)} \\ &= (x+1/x)^3-(x^3+1/x^3) \\ &= 3\left(x+\frac{1}{x}\right) \\ &= 6+3\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2.\end{aligned}$$

It follows that the required minimum value is equal to 6, which is attained at $x=1$. ■

Exercise 2 (RMO 2015f P3)**.** Let

$$N = 2^5+2^{5^2}+2^{5^3}+\cdots+2^{5^{2015}},$$

written in the usual decimal form, find the last two digits of the number N.

Solution 2. Applying Euler's theorem, we obtain [1]

$$2^{20} \equiv 2^{\varphi(25)} \equiv 1 \bmod 25.$$

Note that for any integer $n \geq 1$,

$$5^n-5 = 5(5^{n-1}-1) = 5((1+4)^{n-1}-1)$$

[1] Alternatively, one may observe that $2^{10} = 1024 \equiv -1 \bmod 25$, which yields $2^{20} \equiv 2^{\varphi(25)} \equiv 1 \bmod 25$.

holds, and hence we obtain $5^n \equiv 5 \bmod 20$. This yields

$$N = 2^5 \left(\sum_{i=1}^{2015} 2^{5^i - 5} \right) = 2^5 \left(\sum_{i=1}^{2015} (2^{20})^{\frac{5^i - 5}{20}} \right),$$

and consequently, we obtain

$$N \equiv 2^5 \cdot 2015 \equiv 7 \cdot 15 \equiv 5 \bmod 25.$$

Also note that $N \equiv 0 \bmod 4$ holds. It follows that $N - 80$ is divisible by the relatively prime integers $4, 25$, which shows that 100 divides $N - 80$. Hence, the last two digits of N are $8, 0$. ■

Exercise 3 (RMO 2015f P5)**.** Solve the equation $y^3 + 3y^2 + 3y = x^3 + 5x^2 - 19x + 20$ for positive integers x, y.

Solution 3. Let x, y be positive integers satisfying the above equation. Note that the given equation can be rewritten as

$$(y+1)^3 = x^3 + 5x^2 - 19x + 21.$$

Note that

$$5x^2 - 19x + 21 > 5x^2 - 19x \geq 0$$

holds if $x \geq 4$. Also note that $5x^2 - 19x + 21 > 0$ if x lies in $\{1, 2, 3\}$. Since x is a positive integer, it follows that $5x^2 - 19x + 21 \geq 1$. This shows that $(y+1)^3 > x^3$, and hence, $y + 1 = x + k$ for some positive integer k. Note that

$$(y+1)^3 = x^3 + 5x^2 - 19x + 21$$

is equivalent to

$$3kx^2 + 3k^2x + k^3 = 5x^2 - 19x + 21,$$

which simplifies to

$$(3k - 5)x^2 + (3k^2 + 19)x + (k^3 - 21) = 0.$$

Note that if $k \geq 2$, then using $x > 0$, we obtain

$$\begin{aligned} &(3k - 5)x^2 + (3k^2 + 19)x + (k^3 - 21) \\ &= (3k - 5)x^2 + 3k^2x + 19(x - 1) + k^3 - 2 \\ &> 1. \end{aligned}$$

This shows that k is equal to 0 or 1. If $k = 0$, then $5x^2 - 19x + 21 = 0$ holds, which implies that 21 is even, which is impossible. This shows that $k = 1$, and hence $x = y$. It follows that

$$2x^2 - 22x + 20 = 0,$$

and hence x is equal to one of $1, 10$. Consequently, we obtain (x, y) is equal to $(1, 1)$ or $(10, 10)$.

Note that for any integers x, y, k satisfying $y + 1 = x + k$ and

$$3kx^2 + 3k^2x + k^3 = 5x^2 - 19x + 21,$$

we have

$$(y + 1)^3 = x^3 + 5x^2 - 19x + 21.$$

It follows that $(1, 1), (10, 10)$ also satisfy the given equation.

Consequently, the solutions of the given equation over the positive integers are precisely $(1, 1)$ and $(10, 10)$. ■

Exercise 4 (RMO 2015f P6)**.** From the list of natural numbers $1, 2, 3, \ldots$, suppose we remove all multiples of 7, 11 and 13.

- At which position in the resulting list does the number 1002 appear?
- What number occurs in the position 3600?

Solution 4. Let S denote the set of all positive integers none of which is divisible by 7, 11 or 13. Note that the sum of $\#\{x \in S \mid x \leq 1002\}$ and the number of positive integers ≤ 1002 divisible by 7, 11 or 13 is equal to 1002. Moreover, by the inclusion-exclusion principle, the number of positive integers ≤ 1002 divisible by 7, 11 or 13 is equal to

$$\left\lfloor \frac{1002}{7} \right\rfloor + \left\lfloor \frac{1002}{11} \right\rfloor + \left\lfloor \frac{1002}{13} \right\rfloor - \left\lfloor \frac{1002}{7 \cdot 11} \right\rfloor - \left\lfloor \frac{1002}{11 \cdot 13} \right\rfloor - \left\lfloor \frac{1002}{13 \cdot 7} \right\rfloor + \left\lfloor \frac{1002}{7 \cdot 11 \cdot 13} \right\rfloor$$
$$= 143 + 91 + 77 - 13 - 7 - 11 + 1 = 281.$$

Note that 1002 belongs to S and hence it appears at the $1002 - 281 = 721$st position.

Suppose n occurs at the 3600th position. So the number of positive integers $\leq n$ divisible by 7, 11 or 13 is equal to $n - 3600$. Using the inclusion-exclusion principle, we obtain

$$n - 3600 = \left\lfloor \frac{n}{7} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor - \left\lfloor \frac{n}{7 \cdot 11} \right\rfloor - \left\lfloor \frac{n}{11 \cdot 13} \right\rfloor - \left\lfloor \frac{n}{13 \cdot 7} \right\rfloor + \left\lfloor \frac{n}{7 \cdot 11 \cdot 13} \right\rfloor. \tag{1}$$

This gives

$$\begin{aligned} &\frac{n}{7} - 1 + \frac{n}{11} - 1 + \frac{n}{13} - 1 - \frac{n}{7 \cdot 11} - \frac{n}{11 \cdot 13} - \frac{n}{13 \cdot 7} + \frac{n}{7 \cdot 11 \cdot 13} - 1 \\ &\leq n - 3600 \\ &\leq \frac{n}{7} + \frac{n}{11} + \frac{n}{13} - \frac{n}{7 \cdot 11} + 1 - \frac{n}{11 \cdot 13} + 1 - \frac{n}{13 \cdot 7} + 1 + \frac{n}{7 \cdot 11 \cdot 13}, \end{aligned}$$

which is equivalent to

$$\begin{aligned}&-4\\&\leq n\left(1-\frac{1}{7}-\frac{1}{11}-\frac{1}{13}+\frac{1}{7\cdot 11}+\frac{1}{11\cdot 13}+\frac{1}{13\cdot 7}-\frac{1}{7\cdot 11\cdot 13}\right)-3600\\&\leq 3.\end{aligned}$$

Noting that

$$\left(1-\frac{1}{7}-\frac{1}{11}-\frac{1}{13}+\frac{1}{7\cdot 11}+\frac{1}{11\cdot 13}+\frac{1}{13\cdot 7}-\frac{1}{7\cdot 11\cdot 13}\right)=\frac{6\cdot 10\cdot 12}{7\cdot 11\cdot 13},$$

we obtain

$$-4\times\frac{7\cdot 11\cdot 13}{6\cdot 10\cdot 12}\leq n-5\cdot 7\cdot 11\cdot 13\leq 3\times\frac{7\cdot 11\cdot 13}{6\cdot 10\cdot 12},$$

which yields

$$-\frac{7\cdot 11\cdot 13}{180}\leq n-5\cdot 7\cdot 11\cdot 13\leq\frac{7\cdot 11\cdot 13}{240},$$

and consequently, any solution of the Eq. (1) satisfies

$$-5\leq n-5\cdot 7\cdot 11\cdot 13\leq 4.$$

Note that $n=5\cdot 7\cdot 11\cdot 13$ is the unique solution to

$$n\left(1-\frac{1}{7}-\frac{1}{11}-\frac{1}{13}+\frac{1}{7\cdot 11}+\frac{1}{11\cdot 13}+\frac{1}{13\cdot 7}-\frac{1}{7\cdot 11\cdot 13}\right)-3600=0,$$

and it is a multiple of the pairwise coprime integers $7, 11, 13$. Hence, it is a solution to Eq. (1). Also note that $5\cdot 7\cdot 11\cdot 13-1$ is also a solution to Eq. (1). Moreover, no integer lying in $[-6+5\cdot 7\cdot 11\cdot 13, 4+5\cdot 7\cdot 11\cdot 13]$, other than $5\cdot 7\cdot 11\cdot 13-1$ and $5\cdot 7\cdot 11\cdot 13$ is a solution to Eq. (1). Observe that $5\cdot 7\cdot 11\cdot 13-1$ lies in S.

We conclude that the number that occurs in the 3600th position is $5\cdot 7\cdot 11\cdot 13-1=5004$. ■