RMO 2015e

CRMO 1

Exercise 1 (RMO 2015e P2,). Let $P_1(x) = x^2 + a_1x + b_1$ and $P_2(x) = x^2 + a_2x + b_2$ be two quadratic polynomials with integer coefficients. Suppose $a_1 \neq a_2$ and there exist integers $m \neq n$ such that $P_1(m) = P_2(n), P_2(m) = P_1(n)$. Prove that $a_1 - a_2$ is even.

Solution 1. Using $P_1(m) = P_2(n)$, we get

$$m^2 + a_1m + b_1 = n^2 + a_2n + b_2,$$

that is,

$$(m^2 - n^2) + (a_1m - a_2n) + b_1 - b_2 = 0.$$

Similarly, using $P_1(n) = P_2(m)$, we get

$$(n^2 - m^2) + (a_1n - a_2m) + b_1 - b_2 = 0.$$

This yields

$$2(m^2 - n^2) + (a_1 + a_2)(m - n) = 0.$$

Since $m \neq n$, we get $2(m + n) + a_1 + a_2 = 0$. It follows that $a_1 + a_2$ is even, and hence, so is $a_1 + a_2 - 2a_2 = a_1 - a_2$. ■

Exercise 2 (RMO 2015e P3,). Find all fractions which can be written simultaneously in the forms

$$\frac{7k - 5}{5k - 3} \quad \text{and} \quad \frac{6\ell - 1}{4\ell - 3}$$

for some integers k, ℓ.

Solution 2. The solution relies on the following claim.

> **Claim** — Suppose k, ℓ are integers. Then the equality
>
> $$\frac{7k - 5}{5k - 3} = \frac{6\ell - 1}{4\ell - 3}$$

is equivalent to the pair (k, ℓ) being equal to one of

$$(0,6),(1,-1),(6,-6),(13,-7),(-2,-22),(-3,-15),(-8,-10),(-15,-9). \tag{1}$$

Proof of the Claim. Suppose k, ℓ are integers. Observing that $5k-3$ and $4\ell-3$ are nonzero, it follows that

$$\begin{aligned}
& \frac{7k-5}{5k-3} = \frac{6\ell-1}{4\ell-3} \\
\iff \quad & (7k-5)(4\ell-3) = (5k-3)(6\ell-1) \\
\iff \quad & 28k\ell - 20\ell - 21k + 15 = 30k\ell - 18\ell - 5k + 3 \\
\iff \quad & 2k\ell + 2\ell + 16k - 12 = 0 \\
\iff \quad & k\ell + \ell + 8k - 6 = 0 \\
\iff \quad & (k+1)(\ell+8) = 14.
\end{aligned}$$

This implies that $k+1$ is equal to

$$\pm 1, \pm 2, \pm 7, \pm 14,$$

i.e. k is equal to

$$0, 1, 6, 13, -2, -3, -8, -15. \tag{2}$$

It follows that

$$(k+1)(\ell+8) = 14$$

is equivalent to (k, ℓ) being equal to one of the pairs as in Eq. (1). This proves the Claim. □

Note that if a fraction can be written simultaneously in the forms

$$\frac{7k-5}{5k-3} \quad \text{and} \quad \frac{6\ell-1}{4\ell-3}$$

for two integers k, ℓ, then the Claim implies that (k, ℓ) is equal to the pairs as in Eq. (1), and then k is equal to the integers as in Eq. (2), and consequently, the fraction $\frac{7k-5}{5k-3}$, which is equal to $\frac{6\ell-1}{4\ell-3}$ (by the Claim again), is also equal to

$$\frac{5}{3}, 1, \frac{37}{27}, \frac{43}{31}, \frac{19}{13}, \frac{13}{9}, \frac{61}{43}, \frac{30}{19}. \tag{3}$$

Further[1], observe that the preceeding argument also proves that these fractions

[1] Note that the argument needs to go on since what we have proved so far does not complete the solution. The previous step only says that if a fraction can be written simultaneously in the forms as stated above (and **a priori, it is not clear if there is even a single fraction that can be expressed simultaneously in the stated forms**), then the fraction cannot be anything other than

$$\frac{5}{3}, 1, \frac{37}{27}, \frac{43}{31}, \frac{19}{13}, \frac{13}{9}, \frac{61}{43}, \frac{30}{19}.$$

This does not guarantee if any of these fractions enjoy the stated property.

If this causes any confusion, then it would be a good idea to go through [**Che25**, Chapter 1].

can be written simultaneously in the forms as stated above. Indeed, if (k, ℓ) is one of the pairs as in Eq. (1), and then k is equal to the integers as in Eq. (2), and consequently, the fraction $\frac{7k-5}{5k-3}$, which is equal to $\frac{6\ell-1}{4\ell-3}$ (by the Claim), is also equal to the fractions as in Eq. (3).

We conclude that the fractions as in Eq. (3) are precisely all the fractions with the required property. ■

Exercise 3 (RMO 2015e P4, , cf. RMO 2015a P4, RMO 2015b P4, RMO 2015d P4)**.** Suppose 28 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Solution 3. Let m be an even number. Fix a regular polygon P with m vertices. Then three vertices can be chosen in $\binom{m}{3}$ ways. Note that exactly m of these choices contain two pairs of adjacent vertices, and exactly $m(m-4)$ of these choices contain only a single pair of adjacent vertices. This shows that three vertices of P can be chosen in $\binom{m}{3} - m - m(m-4)$ ways so that no two of the chosen vertices are adjacent. Note that among these type of choices of three vertices, exactly $\frac{m}{2}(m-6)$ contain a pair of diametrically opposite vertices. So the required number is equal to

$$\begin{aligned}
&\binom{m}{3} - m - m(m-4) - \frac{m}{2}(m-6) \\
&= \binom{m}{3} - m(m-3) - \frac{m}{2}(m-6) \\
&= \frac{m}{6}(m^2 - 12m + 38).
\end{aligned}$$

■

Exercise 4 (RMO 2015e P6, , cf. RMO 2015d P6)**.** Find all real numbers a such that $3 < a < 4$ and $a(a - 3\{a\})$ is an integer. (Here, $\{a\}$ is the fractional part of a. For example, $\{1.5\} = 0.5, \{-3.4\} = 1 - 0.4 = 0.6$.)

Solution 4. In the following, k denotes one of the integers $3, 4$.

Let a be a real number such that $k < a < k+1$ and $a(a-3\{a\})$ is an integer. Denote by x the fractional part of a, that is, $x = a - k$. So $a(a-3\{a\})$ is equal to $(k+x)(k-2x) = k^2 - kx - 2x^2$, which is an integer. Thus $2x^2 + kx = \ell$ for some integer ℓ. Since $0 < x < 1$, it follows that $0 < \ell < k+2$. Note that x is equal to

$$\frac{1}{4}(-k \pm \sqrt{k^2 + 8\ell}).$$

Since x is positive, it is equal to $\frac{1}{4}(-k + \sqrt{k^2 + 8\ell})$. It follows that a is equal to $\frac{1}{4}(3k + \sqrt{k^2 + 8\ell})$, for some integer ℓ satisfying $0 < \ell < k + 2$.

Now consider a real number b, which is equal to $\frac{1}{4}(3k+\sqrt{k^2+8\ell})$ for some integer ℓ satisfying $0<\ell<k+2$. Then b lies strictly between k and $k+1$. Put $y=b-k$. It follows from the above that y is a root of

$$2y^2+ky-\ell=0.$$

Note that

$$\begin{aligned} b(b-3\{b\}) &= (k+y)(k-2y) \\ &= k^2-ky-2y^2 \\ &= k^2-\ell, \end{aligned}$$

which is an integer.

This proves that the real numbers lying in the interval $(k,k+1)$ such that $a(a-3\{a\})$ is an integer are precisely those of the form

$$\frac{1}{4}(3k+\sqrt{k^2+8\ell}),$$

where ℓ runs through the integers satisfying $0<\ell<k+2$.

For $k=3$, the real numbers lying in the interval $(3,4)$ satisfying the required condition are precisely

$$\frac{9+\sqrt{17}}{4}, \frac{7}{2}, \frac{9+\sqrt{33}}{4}, \frac{9+\sqrt{41}}{4}.$$

For $k=4$, the real numbers lying in the interval $(4,5)$ satisfying the required condition are precisely

$$3+\sqrt{3/2}, 3+\sqrt{2}, 3+\sqrt{5/2}, 3+\sqrt{3}, 3+\sqrt{7/2}.$$

■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 2)