RMO 2015d

CRMO 2 (Odisha)

Exercise 1 (RMO 2015d P2,). Let $P(x) = x^2 + ax + b$ be a quadratic polynomial with real coefficients. Suppose there are real numbers s is not equal to t such that $P(s) = t$ and $P(t) = s$. Prove that $b - st$ is a root of the equation $x^2 + ax + b - st = 0$.

Solution 1. We have

$$s^2 + as + b = t, t^2 + at + b = s.$$

Taking their difference, we obtain $(s - t)(s + t + a + 1) = 0$, which gives $s + t + a + 1 = 0$ since $s \neq t$. Using the above, we obtain

$$s(s^2 + as + b) - t(t^2 + at + b) = 0,$$

or equivalently,

$$(s - t)(b + a(s + t) + s^2 + st + t^2) = 0.$$

Combining the above with $s + t + a + 1 = 0$ and $s \neq t$, we obtain

$$b - (s + t) - st = 0.$$

Note that

$$\begin{aligned}(b - st)^2 + a(b - st) + b - st &= (b - st)(b - st + a + 1)\\ &= (b - st)(s + t + a + 1)\\ &= 0.\end{aligned}$$

This completes the proof. ■

Exercise 2 (RMO 2015d P3,). Find all integers a, b, c such that $a^2 = bc+1$ and $b^2 = ca + 1$.

Solution 2. Let a, b, c be integers satisfying the given equations.

Let us first consider the case that $a = b$. Note that $a(a - c) = 1$ holds, which shows that $(a, a - c)$ is equal to one of $(1, 1), (-1, -1)$, which implies that (a, b, c) is equal to one of $(1, 1, 0), (-1, -1, 0)$.

Now, let us consider the case that $a \neq b$. Taking the difference of the given equations, we obtain $(a-b)(a+b+c) = 0$, which gives $a+b = -c$. Substituting $c = -a - b$ in $a^2 = bc + 1$ yields $a^2 + ab + b^2 = 1$, which is equivalent to

$$(2a + b)^2 + 3b^2 = 4.$$

This shows that $(2a + b, b)$ is equal to one of

$$(1, 1), (1, -1), (-1, 1), (-1, -1), (2, 0), (-2, 0).$$

This implies that (a, b, c) is equal to one of

$$(0, 1, -1), (1, -1, 0), (-1, 1, 0), (0, -1, 1), (1, 0, -1), (-1, 0, 1).$$

Combining the above cases, it follows that (a, b, c) is equal to one of

$$(1, 1, 0), (-1, -1, 0), (0, 1, -1), (1, -1, 0), (-1, 1, 0), (0, -1, 1), (1, 0, -1), (-1, 0, 1).$$

Note that any of the above triples satisfies the given equations. Hence, the solutions of the given equations over the integers are precisely the above eight triples. ■

Exercise 3 (RMO 2015d P4, , cf. RMO 2015a P4, RMO 2015b P4, RMO 2015e P4)**.** Suppose 32 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Solution 3. Let m be an even number. Fix a regular polygon P with m vertices. Then three vertices can be chosen in $\binom{m}{3}$ ways. Note that exactly m of these choices contain two pairs of adjacent vertices, and exactly $m(m-4)$ of these choices contain only a single pair of adjacent vertices. This shows that three vertices of P can be chosen in $\binom{m}{3} - m - m(m-4)$ ways so that no two of the chosen vertices are adjacent. Note that among these type of choices of three vertices, exactly $\frac{m}{2}(m-6)$ contain a pair of diametrically opposite vertices. So the required number is equal to

$$\begin{aligned}
&\binom{m}{3} - m - m(m-4) - \frac{m}{2}(m-6) \\
&= \binom{m}{3} - m(m-3) - \frac{m}{2}(m-6) \\
&= \frac{m}{6}(m^2 - 12m + 38).
\end{aligned}$$

■

The content posted here and at this blog by Evan Chen are quite useful.

Exercise 4 (RMO 2015d P6, , cf. RMO 2015e P6)**.** Find all real numbers a such that $4 < a < 5$ and $a(a-3\{a\})$ is an integer. (Here, $\{a\}$ is the fractional part of a. For example, $\{1.5\} = 0.5, \{-3.4\} = 1 - 0.4 = 0.6$.)

Solution 4. In the following, k denotes one of the integers $3, 4$.

Let a be a real number such that $k < a < k+1$ and $a(a-3\{a\})$ is an integer. Denote by x the fractional part of a, that is, $x = a - k$. So $a(a-3\{a\})$ is equal to $(k+x)(k-2x) = k^2 - kx - 2x^2$, which is an integer. Thus $2x^2 + kx = \ell$ for some integer ℓ. Since $0 < x < 1$, it follows that $0 < \ell < k+2$. Note that x is equal to

$$\frac{1}{4}(-k \pm \sqrt{k^2 + 8\ell}).$$

Since x is positive, it is equal to $\frac{1}{4}(-k + \sqrt{k^2 + 8\ell})$. It follows that a is equal to $\frac{1}{4}(3k + \sqrt{k^2 + 8\ell})$, for some integer ℓ satisfying $0 < \ell < k+2$.

Now consider a real number b, which is equal to $\frac{1}{4}(3k + \sqrt{k^2 + 8\ell})$ for some integer ℓ satisfying $0 < \ell < k+2$. Then b lies strictly between k and $k+1$. Put $y = b - k$. It follows from the above that y is a root of

$$2y^2 + ky - \ell = 0.$$

Note that

$$\begin{aligned} b(b-3\{b\}) &= (k+y)(k-2y) \\ &= k^2 - ky - 2y^2 \\ &= k^2 - \ell, \end{aligned}$$

which is an integer.

This proves that the real numbers lying in the interval $(k, k+1)$ such that $a(a-3\{a\})$ is an integer are precisely those of the form

$$\frac{1}{4}(3k + \sqrt{k^2 + 8\ell}),$$

where ℓ runs through the integers satisfying $0 < \ell < k+2$.

For $k = 3$, the real numbers lying in the interval $(3, 4)$ satisfying the required condition are precisely

$$\frac{9+\sqrt{17}}{4}, \frac{7}{2}, \frac{9+\sqrt{33}}{4}, \frac{9+\sqrt{41}}{4}.$$

For $k = 4$, the real numbers lying in the interval $(4, 5)$ satisfying the required condition are precisely

$$3 + \sqrt{3/2}, 3 + \sqrt{2}, 3 + \sqrt{5/2}, 3 + \sqrt{3}, 3 + \sqrt{7/2}.$$

■