RMO 2015c

Mumbai

Exercise 1 (RMO 2015c P2,). Determine the number of 3 digit numbers in base 10 having at least one digit equal to 5 and at most one digit equal to 3.

Solution 1. Note that the number of three digit numbers satisfying the given conditions is equal to the number ordered triples (a, b, c) such that $a \neq 0$, $0 \leq a, b, c \leq 9$, at least one of a, b, c is equal to 5 and at most one of them is equal to 3. The number of triples (a, b, c) with $a \neq 0$, $0 \leq a, b, c \leq 9$ is equal to $9 \times 10 \times 10 = 900$. Note that the number of triples (a, b, c) such that $a \neq 0$, $a, b, c \in \{0, 1, 2, 3, 4, 6, 7, 8, 9\}$ is equal to $8 \times 9 \times 9 = 9 \times 72$. So there are $900 - 9 \times 72 = 9 \times 28 = 252$ triples (a, b, c) such that $a \neq 0$, $0 \leq a, b, c \leq 9$, at least one of a, b, c is equal to 5. Note that among such triples, there are three triples which contain more than one 3 (namely, $(5, 3, 3), (3, 5, 3), (3, 3, 5)$). So the required number is $252 - 3 = 249$. ■

Solution 2. Let A_0 denote the number of three digit numbers with no digit equal to 3, let A_1 denote the number of three digit numbers with exactly one digit equal to 3, and let B denote the number of three digit numbers with no digit equal to 5. We need to determine the size of the set $B^c \cap (A_0 \cup A_1)$, where B^c denotes the complement of B in the set of three digit numbers. Note that

$$\begin{aligned} |B^c \cap (A_0 \cup A_1)| &= |A_0 \cup A_1| - |(A_0 \cup A_1) \cap B| \\ &= |A_0| + |A_1| - |A_0 \cap B| - |A_1 \cap B|. \end{aligned}$$

Observe that

$$\begin{aligned} |A_0| &= 8 \cdot 9 \cdot 9, \\ |A_1| &= 9 \cdot 9 + 8 \cdot 9 + 8 \cdot 9, \end{aligned}$$

and similarly,

$$\begin{aligned} |A_0 \cap B| &= 7 \cdot 8 \cdot 8, \\ |A_1 \cap B| &= 8 \cdot 8 + 7 \cdot 8 + 7 \cdot 8. \end{aligned}$$

This yields

$$|B^c \cap (A_0 \cup A_1)| = 9 \cdot 9 \cdot 9 + 2 \cdot 8 \cdot 9 - 8 \cdot 8 \cdot 8 - 2 \cdot 7 \cdot 8$$

$$= 2 \cdot 2 \cdot 8 + 81 + 72 + 64$$
$$= 249.$$

■

Exercise 2 (RMO 2015c P3,). Let $P(x)$ be a nonconstant polynomial whose coefficients are positive integers. If $P(n)$ divides $P(P(n) - 2015)$ for all natural numbers n, then prove that $P(-2015) = 0$.

Summary — In absolute value, a higher degree polynomial dominates a smaller degree polynomial at arguments which are large enough in absolute value.

Solution 3. Let $Q(x), R(x)$ be polynomials with rational coefficients such that

$$P(P(x) - 2015) = P(x)Q(x) + R(x)$$

and $R(x) = 0$ or $\deg R(x) < \deg P(x)$. Note that $P(n)$ is positive for all integer $n \geq 1$ since the coefficients of $P(x)$ are positive integers. By the given condition, it follows that $P(n)$ divides $R(n)$ for any integer $n \geq 1$.

Claim — Let $f(x), g(x)$ be two polynomials with real coefficients. Suppose $f(x)$ is a nonconstant polynomial with a positive leading coefficient, and $\deg g(x) < \deg f(x)$. Then there exists an integer $n_0 \geq 1$ such that

$$f(n) > g(n)$$

for any $n \geq n_0$.

Proof of the Claim. Note that it suffices to proves the Claim if $f(x)$ is a monomial, that is, a power of x. Indeed, write $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$ with $a_0, \ldots, a_d \in \mathbb{R}$ and d denoting the degree of f. Also write $g(x) = b_e x^e + b_{e-1}x^{e-1} + \cdots + b_0$ with $b_0, \ldots, b_e \in \mathbb{R}$ and e denoting the degree of g. Noting that $a_d > 0$, it follows that for a positive integer n, the inequality

$$a_d n^d + a_{d-1}n^{d-1} + \cdots + a_0 > b_e n^e + b_{e-1}n^{e-1} + \cdots + b_0$$

holds if

$$a_d n^d > \frac{a_{d-1}}{a_d} n^{d-1} + \cdots + \frac{a_0}{a_d} + \frac{b_e}{a_d} n^e + \frac{b_{e-1}}{a_d} n^{e-1} + \cdots + \frac{b_0}{a_d}$$

is satisfied, which can be concluded provided the Claim is known in the case when f is a monomial.

The content posted here and at this blog by Evan Chen are quite useful.

Let us assume that f is a monomial. Write $f(x) = x^d$ where d denotes the degree of f, and write $g(x) = b_e x^e + b_{e-1}x^{e-1} + \cdots + b_0$ with $b_0, \ldots, b_e \in \mathbb{R}$ and e denoting the degree of g. For any integer n, note that

$$\begin{aligned} f(n) - g(n) &= \left(\frac{1}{e+1}n^d - b_e n^e\right) \\ &\quad + \left(\frac{1}{e+1}n^d - b_{e-1}n^{e-1}\right) \\ &\quad + \cdots + \left(\frac{1}{e+1}n^d - b_0\right) \\ &\geq \left(\frac{1}{e+1}n^d - |b_e| n^e\right) \\ &\quad + \left(\frac{1}{e+1}n^d - |b_{e-1}| n^{e-1}\right) \\ &\quad + \cdots + \left(\frac{1}{e+1}n^d - |b_0|\right). \end{aligned}$$

Since $d \geq e$, it follows that there exists an integer $n_0 \geq 1$ such that

$$\frac{1}{e+1}n^d - |b_e| n^e \frac{1}{e+1}n^d - |b_{e-1}| n^{e-1}, \ldots, \frac{1}{e+1}n^d - |b_0|$$

are positive for any $n \geq n_0$. This proves the Claim. □

By the above Claim, it follows that $R(x)$ is the zero polynomial. This implies that

$$P(P(x) - 2015) = P(x)Q(x).$$

Since $P(x)$ is a nonconstant polynomial, it has a root z in $\mathbb{C}$. Substituting $x = z$ yields

$$P(-2015) = 0.$$

■

Here is an alternate solution.

Solution 4. Since $P(x)$ is a polynomial with integer coefficients, it follows that for any integers a, b, the difference $P(a) - P(b)$ is divisible by $a - b$. This shows that

$$P(P(n) - 2015) - P(-2015)$$

is divisible by $P(n)$ for all natural numbers n. Since $P(n)$ divides $P(P(n)-2015)$ for all natural numbers n, it follows that $P(n)$ divides $P(-2015)$ for all natural numbers n. Let d denote the degree of $P(x)$, and a_d denote the leading coefficient of $P(x)$. Since $P(x)$ has positive integer coefficients, it follows that

$$P(n) \geq a_d n^d$$

holds for any natural number n. Taking $k = |P(-2015)|$, we have that $P(k+1)$ divides k. Since $P(k+1) \geq a_d(k+1)^d > k$ and k is a nonnegative integer, we conclude that $k = 0$, which implies that $P(-2015) = 0$. ■

Exercise 3 (RMO 2015c P4,). Find all three digit natural numbers of the form $(abc)_{10}$ such that $(abc)_{10}$, $(bca)_{10}$, $(cab)_{10}$ are in geometric progression. (Here $(abc)_{10}$ is representation in base 10).

Solution 5. Let $(abc)_{10}$ be a three digit natural number such that $(abc)_{10}$, $(bca)_{10}$, $(cab)_{10}$ are in geometric progression, that is,

$$(100b + 10c + a)^2 = (100a + 10b + c)(100c + 10a + b)$$

holds, which is equivalent to

$$\begin{aligned} &10000b^2 + 100c^2 + a^2 + 2000bc + 200ab + 20ca \\ &= 10000ca + 1000(a^2 + bc) + 100(c^2 + 2ab) + 10(b^2 + ca) + bc. \end{aligned}$$

This implies

$$10000(b^2 - ca) - 1000(a^2 - bc) - 10(b^2 - ca) + (a^2 - bc) = 0,$$

which reduces to

$$(10(b^2 - ca) - (a^2 - bc))(1000 - 1) = 0.$$

This gives

$$(10a - b)c = 10b^2 - a^2. \tag{1}$$

Note that $10a - b$ is nonzero, otherwise, 10 would divide b, implying $b = 0$, and thus, $a = 0$, which is impossible. It follows that

$$\begin{aligned} c &= \frac{10b^2 - a^2}{10a - b} \\ &= \frac{10b(b - 10a) + 100ab - a^2}{10a - b} \\ &= -10b + \frac{100ab - a^2}{10a - b} \\ &= -10b + a\frac{100b - a}{10a - b} \\ &= -10b + a\frac{100(b - 10a) + 1000a - a}{10a - b} \\ &= -10b - 100a + \frac{999a^2}{10a - b}. \end{aligned}$$

Let d denote the greatest common divisor of a, b. Note that $10a - b$ divides $999a^2$, which shows that $10\frac{a}{d} - \frac{b}{d}$ divides $999d\left(\frac{a}{d}\right)^2$. Since $a/d, b/d$ are relatively

prime, it follows that $10\frac{a}{d} - \frac{b}{d}$ divides $999d$. Using Eq. (1), it follows that $b \geq 1$. Note that if $a = b$ holds, then using Eq. (1), we obtain $a = b = c$, and hence, $(abc)_{10}$, $(bca)_{10}$, $(cab)_{10}$ are equal, and form a geometric progression. It remains to consider the case that $a \neq b$, which we assume from now on. Since $1 \leq b \leq 9$ and d divides b, it follows that $d \leq 4$.

Let us consider the case that $d = 4$. Then $a/d, b/d$ belong to $\{1, 2\}$. Using $a \neq b$, it follows that $(\frac{a}{d}, \frac{b}{d})$ is equal to $(1, 2)$ or $(2, 1)$, which implies that $10\frac{a}{d} - \frac{b}{d}$ does not divide $999d$, which is impossible. We obtain $d \neq 4$.

Let us consider the case that $d = 3$. Note that $1 \leq a/d \leq 3$, $1 \leq b/d \leq 3$ hold, and hence $7 = 10-3 \leq 10\frac{a}{d} - \frac{b}{d} \leq 30-1 = 29 < 37$. Since $10\frac{a}{d} - \frac{b}{d}$ divides $999d = 37 \times 81$, it is equal to one of $9, 27$, which shows that $\frac{a}{d} - \frac{b}{d} \equiv 0 \bmod 3$. This yields $a = b$, which is impossible. We get that $d \neq 3$.

Let us consider the case that $d = 2$. Note that $1 \leq a/d \leq 4, 1 \leq b/d \leq 4$ hold, which yields $6 \leq 10a/d - b/d \leq 39$. Since $10a/d - b/d$ divides $999d = 2 \times 27 \times 37$, it is equal to one of $27, 37$, and it follows that $(a/d, b/d)$ is equal to one of $(3, 3), (4, 3)$. Using $a \neq b$, we get that (a, b) is equal to $(8, 6)$. Then Eq. (1) yields $c = 4$.

Let us consider the case that $d = 1$. Note that $10a - b$ divides $999 = 27 \times 37$. Observing that $10a - b \leq 90$, we obtain $10a - b$ is equal to one of $1, 3, 9, 27, 37$, and hence (a, b) is equal to one of $(1, 9), (1, 7), (1, 1), (3, 3), (4, 3)$. Note that (a, b) is equal to none of $(1, 9), (1, 7)$, otherwise, we would get $c > 10$. Using $a \neq b$, it follows that (a, b) is equal to $(4, 3)$. Then Eq. (1) yields $c = 2$.

The above argument shows that $(abc)_{10}$ is equal to one of the following eleven integers

$$111, 222, 333, 444, 555, 666, 777, 888, 999, 432, 864.$$

Note that

$$\begin{aligned} 432 \times 243 &= 144 \times 729 \\ &= 12^2 \times 27^2 \\ &= 324^2, \\ 864 \times 486 &= 4 \times 432 \times 243 \\ &= 4 \times 324^2 \\ &= 648^2 \end{aligned}$$

holds. This proves that the above eleven integers satisfy the required condition, and hence, these are precisely all the required three digit natural numbers. ■

Exercise 4 (RMO 2015c P6,). Let $S = \{1, 2, \ldots, n\}$ and let T be the set of all ordered triples of subsets of S, say (A_1, A_2, A_3), such that $A_1 \cup A_2 \cup A_3 = S$. Determine in terms of n,

$$\sum_{(A_1, A_2, A_3) \in T} |A_1 \cap A_2 \cap A_3|,$$

where $|X|$ denotes the number of elements in the set X.

> **Walkthrough** — Find a suitable one-to-one correspondence between the triples of the subsets of S, whose union is S and intersection is a given subset X of S, and the maps from $S \setminus X$ to $\{1, 2, \ldots, 6\}$.

Solution 6. Note that for $0 \leq r \leq n$, the set S has $\binom{n}{r}$ many subsets with r elements. Also note that for $0 \leq r < n$, and given a subset X of S with r elements, the number of ordered tuples (A_1, A_2, A_3) of subsets of S with $A_1 \cap A_2 \cap A_3 = X$ and $A_1 \cup A_2 \cup A_3 = S$ is equal to the number of maps from $S \setminus X$ to $\{1, 2, 3, 4, 5, 6\}$, which is equal to 6^{n-r}. Moreover, there is only one ordered tuples (A_1, A_2, A_3) of subsets of S with $A_1 \cap A_2 \cap A_3 = S$. So

$$\begin{aligned}\sum_{(A_1,A_2,A_3)\in T} |A_1 \cap A_2 \cap A_3| &= \sum_{r=0}^{n-1} \binom{n}{r} 6^{n-r} r + n \\ &= \sum_{r=1}^{n} \binom{n}{r} 6^{n-r} r \\ &= n \sum_{r=0}^{n-1} \binom{n-1}{r} 6^{n-1-r} \\ &= n7^{n-1},\end{aligned}$$

where the final equality is obtained by applying the binomial theorem, and the second last equality is obtained by counting the size of the following set in two different ways.

$$\{(A, a, f) \mid A \subseteq S, a \in A, f \text{ is a map from } S \setminus A \text{ to } \{1, 2, 3, 4, 5, 6\}\}$$

■

Exercise 5 (RMO 2015c P7,). Let $x, y, z \in \mathbb{R}$, such that $x^2 + y^2 + z^2 - 2xyz = 1$. Prove that

$$(1 + x)(1 + y)(1 + z) \leq 4 + 4xyz.$$

Solution 7. Note that

$$\begin{aligned}&4 + 4xyz - (1 + x)(1 + y)(1 + z) \\ &= 4 + 4xyz - (1 + x + y + z + x^2 + y^2 + z^2 + xyz) \\ &= 4 + 3xyz - (1 + x + y + z + x^2 + y^2 + z^2) \\ &= 4 + 3\frac{x^2 + y^2 + z^2 - 1}{2} - (1 + x + y + z + x^2 + y^2 + z^2)\end{aligned}$$

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$$= \frac{1}{2}(5 + 3(x^2 + y^2 + z^2) - 2(1 + x + y + z + x^2 + y^2 + z^2))$$
$$= \frac{1}{2}(3 + x^2 + y^2 + z^2 - 2(x + y + z))$$
$$= \frac{1}{2}((x-1)^2 + (y-1)^2 + (z-1)^2),$$

which is nonnegative. So the required inequality follows. ■