RMO 2015b

CRMO 4 (Karnataka)

Exercise 1 (RMO 2015b P2,). Let $P(x) = x^2 + ax + b$ be a quadratic polynomial where a is real and $b \neq 2$, is rational. Suppose $P(0)^2, P(1)^2, P(2)^2$ are integers, prove that a and b are integers.

Solution 1. Since b is rational and $P(0)^2 = b^2$ is an integer, it follows that b is an integer. Note that

$$\begin{aligned} P(1)^2 &= (1 + a + b)^2 \\ &= 1 + a^2 + b^2 + 2a + 2b + 2ab, \\ P(2)^2 &= (4 + 2a + b)^2 \\ &= 16 + 4a^2 + b^2 + 16a + 8b + 4ab. \end{aligned}$$

Since b is an integer, the given conditions imply that $a^2+2a+2ab, 4a^2+16a+4ab$ are integers. This shows that

$$\begin{aligned} 4a^2 + 16a + 4ab - 2(a^2 + 2a + 2ab) &= 2a^2 + 12a, \\ 4a^2 + 16a + 4ab - 4(a^2 + 2a + 2ab) &= 8a - 4ab \end{aligned}$$

are integers. Since $b \neq 2$ and b is an integer, it follows that a is a rational number. Combining this with the fact that $2a^2 + 12a$ is rational, it follows that a is equal to $\frac{n}{2}$ for some integer n. Indeed, write $a = \frac{x}{y}$ where x, y are integers with $y \geq 1$ and $\gcd(x, y) = 1$. Note that $2\frac{x^2}{y} + 12x$ is an integer. Since x and y are relatively prime, this implies that y divides 2. Consequently, a is equal to $\frac{n}{2}$ for some integer n. Using that $2a^2 + 12a$ is an integer, we get that $\frac{n^2}{2}$ is also an integer. This shows that n is even, and hence a is an integer. This completes the proof. ■

Exercise 2 (RMO 2015b P3,). Find all integers a, b, c such that $a^2 = bc+4$ and $b^2 = ca + 4$.

Summary — In absolute value, a higher degree polynomial dominates a smaller degree polynomial at arguments which are large enough in absolute value.

Solution 2. Let a, b, c be integers satisfying the given equations.

Let us first consider the case that $a = b$. Note that $a(a - c) = 4$ holds, which shows that $(a, a - c)$ is equal to one of the elements of

$$\{(d, d - 4/d) \mid d \text{ is a divisor of } 4\},$$

and hence (a, b, c) is equal to one of

$$(1, 1, -3), (-1, -1, 3), (2, 2, 0), (-2, -2, 0), (4, 4, 3), (-4, -4, -3).$$

Now, let us consider the case that $a \neq b$. Note that

$$a^3 - b^3 = a(bc + 4) - b(ca + 4) = 4(a - b)$$

holds, which yields

$$a^2 + ab + b^2 = 4.$$

It follows that at least one of a, b is even, and hence, both of them are even. Observe that

$$\left(a + \frac{b}{2}\right)^2 + 3\left(\frac{b}{2}\right)^2 = 4$$

holds, which shows that $(a + b/2, b/2)$ is equal to one of

$$(2, 0), (-2, 0), (1, 1), (1, -1), (-1, 1), (-1, -1),$$

and hence (a, b) is equal to one of

$$(2, 0), (-2, 0), (0, 2), (2, -2), (-2, 2), (0, -2).$$

This implies that (a, b, c) is equal to one of

$$(2, 0, -2), (-2, 0, 2), (0, 2, -2), (2, -2, 0), (-2, 2, 0), (0, -2, 2).$$

Considering the above cases, it follows that (a, b, c) is equal to one of

$$(1, 1, -3), (-1, -1, 3), (2, 2, 0), (-2, -2, 0), (4, 4, 3), (-4, -4, -3),$$

$$(2, 0, -2), (-2, 0, 2), (0, 2, -2), (2, -2, 0), (-2, 2, 0), (0, -2, 2).$$

Note that any of the above pairs satisfy the given equations. This proves that the above tuples are precisely all the solutions of the given equation over the integers. ■

Exercise 3 (RMO 2015b P4, cf. RMO 2015a P4, RMO 2015d P4, RMO 2015e P4)**.** Suppose 40 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Solution 3. Let m be an even number. Fix a regular polygon P with m vertices. Then three vertices can be chosen in $\binom{m}{3}$ ways. Note that exactly m of these choices contain two pairs of adjacent vertices, and exactly $m(m-4)$ of these choices contain only a single pair of adjacent vertices. This shows that three vertices of P can be chosen in $\binom{m}{3} - m - m(m-4)$ ways so that no two of the chosen vertices are adjacent. Note that among these type of choices of three vertices, exactly $\frac{m}{2}(m-6)$ contain a pair of diametrically opposite vertices. So the required number is equal to

$$\begin{aligned}
&\binom{m}{3} - m - m(m-4) - \frac{m}{2}(m-6) \\
&= \binom{m}{3} - m(m-3) - \frac{m}{2}(m-6) \\
&= \frac{m}{6}(m^2 - 12m + 38).
\end{aligned}$$

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Exercise 4 (RMO 2015b P6)**.** Find the number of integers m that satisfy both the following properties:

1. $1 \leq m \leq 5000$,
2. $[\sqrt{m}] = [\sqrt{m+125}]$.

Solution 4. If m is positive integer satisfying $[\sqrt{m}] = [\sqrt{m+125}]$, then for $k = [\sqrt{m}]$, we have $k \leq \sqrt{m} < \sqrt{m+125} < k+1$, which yields $k^2 \leq m < m+125 < (k+1)^2$, and this implies that

$$2k + 1 = (k+1)^2 - k^2 \geq m + 126 - m = 126,$$

or equivalently, $k \geq 63$ holds. This shows that for any positive integer m satisfying $[\sqrt{m}] = [\sqrt{m+125}]$, we have $k^2 \leq m \leq (k+1)^2 - 126$ for some integer $k \geq 63$.

Conversely, if m is an integer satisfying $k^2 \leq m \leq (k+1)^2 - 126$ for some positive integer $k \geq 63$, then we obtain $k^2 \leq m < m+125 < (k+1)^2$, which gives $k \leq \sqrt{m} < \sqrt{m+125} < k+1$, and consequently, $[\sqrt{m}] = [\sqrt{m+125}]$ holds.

So the number of integers satisfying the given conditions is equal to the number of positive integers m satisfying $m \leq 5000$ and $k^2 \leq m \leq (k+1)^2 - 126$ for some positive integer $k \geq 63$. Note that any such integer k satisfies

$k^2 \leq m \leq 5000 < 71^2$, which gives $k \leq 70$. Moreover, if ℓ is an integer satisfying $63 \leq \ell \leq 70$ and an integer m satisfies $\ell^2 \leq m \leq (\ell+1)^2 - 126$, then using

$$(\ell+1)^2 - 126 \leq 71^2 - 126 = 4900 + 140 + 1 - 126 < 5000,$$

it follows that $m \leq 5000$. This proves that the integers satisfying the given conditions are precisely the integers m satisfying $k^2 \leq m \leq (k+1)^2 - 126$ for some integer $63 \leq k \leq 70$. Hence, the required number is equal to

$$\begin{aligned}
&\sum_{k=63}^{70} ((k+1)^2 - 126 - k^2 + 1) \\
&= \sum_{k=63}^{70} (2k - 124) \\
&= 2 + 4 + 6 + \cdots + 16 \\
&= 72.
\end{aligned}$$

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