RMO 2015a

CRMO 3 (West Bengal)

Exercise 1 (RMO 2015a P2,). Let $P(x) = x^2 + ax + b$ be a quadratic polynomial where a, b are real numbers. Suppose $\langle P(-1)^2, P(0)^2, P(1)^2 \rangle$ is an AP of positive integers. Prove that a, b are integers.

Solution 1. Note that $P(-1)^2, P(0)^2, P(1)^2$ are equal to

$$1 + a^2 + b^2 - 2a + 2b - 2ab, b^2, 1 + a^2 + b^2 + 2a + 2b + 2ab$$

respectively. Since they form an arithmetic progression, we obtain

$$1 + a^2 + b^2 - 2a + 2b - 2ab + 1 + a^2 + b^2 + 2a + 2b + 2ab = 2b^2,$$

or equivalently, $a^2 + 2b + 1 = 0$. It follows that

$$b^2 - 2a - 2ab, b^2, b^2 + 2a + 2ab$$

form an arithmetic progression of positive integers. Note that

$$\begin{aligned}(2a + 2ab)^2 &= 4a^2(b + 1)^2 \\ &= -4(2b + 1)(b + 1)^2 \\ &= -4(2b^3 + 5b^2 + 4b + 1) \\ &= -4((2b^2 + 4)b + 5b^2 + 1).\end{aligned}$$

Since b^2 is an integer, it follows that b is rational number. Since b is rational and b^2 is an integer, it follows that b is an integer. Using $a^2 + 2b + 1 = 0$, it follows that a^2 is an integer. Moreover, if $b = -1$, then a is an integer. If $b \neq -1$, then using that $2a + 2ab$ is an integer, we obtain a is rational. Since a^2 is an integer and a is rational, it follows that a is an integer. This completes the proof. ■

Exercise 2 (RMO 2015a P3, , Canadian Mathematical Olympiad 1991 P1, ??). Show that there are infinitely many triples (x, y, z) of integers, such that $x^3 + y^4 = z^{31}$.

Walkthrough —

(a) Note that

$$2^{12k} + 2^{12k} = 2^{12k+1}$$

holds.

(b) It suffices to find an positive integer k such that $12k + 1 = 31\ell$ holds for some positive integer ℓ. This shows that any such integer ℓ satisfies $\ell \equiv 1 \bmod 3$, and $\ell \equiv -1 \bmod 4$, which implies that $\ell \equiv 7 \bmod 12$. Moreover, if ℓ is congruent to 7 modulo 12, then

$$31\ell - 1 \equiv 31 * 7 - 1 \bmod 12 \equiv 7^2 - 1 \bmod 12 \equiv 0 \bmod 12$$

holds, or equivalently, there exists an integer k_ℓ such that

$$12k_\ell + 1 = 31\ell$$

holds.

(c) Note that

$$(2^{4k_\ell}, 2^{3k_\ell}, 2^\ell)$$

satisfies the given equation, for any positive integer $\ell \equiv 7 \bmod 12$.

Remark. Alternatively, one may argue as follows, motivated by the solution of ??.

Solution 2. Let a, b, c be integers such that $a + b = c$. Multiplying by $a^p b^q c^r$, we get

$$a^{p+1} b^q c^r + a^p b^{q+1} c^r = a^p b^q c^{r+1}.$$

Note that $a^{p+1} b^q c^r, a^p b^{q+1} c^r, a^p b^q c^{r+1}$ are a 3rd, 4th, and a 31st power respectively if 3 divides $p + 1, q, r$, 4 divides $p, q + 1, r$, 31 divides $p, q, r + 1$, which holds if there are positive integers P, Q, R such that

$$p = 4 \cdot 31 \cdot P, q = 3 \cdot 31 \cdot Q, r = 3 \cdot 4 \cdot R$$

holds, and 3 divides $4 \cdot 31 \cdot P + 1$, 4 divides $3 \cdot 31 \cdot Q + 1$, 31 divides $3 \cdot 4 \cdot R + 1$. Since the integers $3, 4, 31$ are pairwise coprime, it follows that each of them is coprime to the product of the remaining two integers, and hence, by the division algorithm, there are positive integers P, Q, R such that 3 divides $4 \cdot 31 \cdot P + 1$, 4 divides $3 \cdot 31 \cdot Q + 1$, 31 divides $3 \cdot 4 \cdot R + 1$, or equivalently,

$$P \equiv 2 \bmod 3, Q \equiv 3 \bmod 4, R \equiv 18 \bmod 31$$

holds. In the above, the last congruence is obtained using $(3 \cdot 4) \cdot (6 \cdot 3) \equiv -1 \bmod 31$, which follows from $3 \cdot 4 \cdot 6 \equiv 10 \bmod 31$ and $10 \cdot 3 \equiv -1 \bmod 31$.

This shows that for any positive integers P, Q, R satisfying the above congruence conditions (for instance, $P = 2, Q = 3, R = 18$), and for any positive integers a, b, c satisfying $a + b = c$, the triple

$$((a^{p+1} b^q c^r)^{\frac{1}{3}}, (a^p b^{q+1} c^r)^{\frac{1}{4}}, (a^p b^q c^{r+1})^{\frac{1}{31}})$$

The content posted here and at this blog by Evan Chen are quite useful.

satisfies the given equation, where $p = 124P, q = 93Q, r = 12R$. Hence the given equation has infinitely many solution in integers. ■

Exercise 3 (RMO 2015a P4, cf. RMO 2015b P4, RMO 2015d P4, RMO 2015e P4)**.** Suppose 36 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Solution 3. Let m be an even number. Fix a regular polygon P with m vertices. Then three vertices can be chosen in $\binom{m}{3}$ ways. Note that exactly m of these choices contain two pairs of adjacent vertices, and exactly $m(m-4)$ of these choices contain only a single pair of adjacent vertices. This shows that three vertices of P can be chosen in $\binom{m}{3} - m - m(m-4)$ ways so that no two of the chosen vertices are adjacent. Note that among these type of choices of three vertices, exactly $\frac{m}{2}(m-6)$ contain a pair of diametrically opposite vertices. So the required number is equal to

$$\begin{aligned}
&\binom{m}{3} - m - m(m-4) - \frac{m}{2}(m-6) \\
&= \binom{m}{3} - m(m-3) - \frac{m}{2}(m-6) \\
&= \frac{m}{6}(m^2 - 12m + 38).
\end{aligned}$$

■

Exercise 4 (RMO 2015a P6,)**.** Show that there are infinitely many positive real numbers a, which are not integers, such that $a(a - 3\{a\})$ is an integer. (Here, $\{a\}$ is the fractional part of a. For example, $\{1.5\} = 0.5, \{-3.4\} = 1 - 0.4 = 0.6$.)

Solution 4. If n is a positive integer and $0 \le x < 1$, then $(n+x)(n+x-3\{n+x\})$ is an integer if and only if $(n+x)(n-2x) = n^2 - nx - 2x^2$ is an integer. Note that if n is odd and $x = \frac{1}{2}$, then $n^2 - nx - 2x^2$ is an integer, and hence, so is $(n+x)(n+x-3\{n+x\})$. It follows that for any rational number a of the form $2k + 1 + \frac{1}{2}$, where k is a nonnegative integer, the real number $a(a - 3\{a\})$ is also an integer. This completes the proof. ■