RMO 2014e

Mumbai

Exercise 1 (RMO 2014e P1,). Three positive real numbers a, b, c are such that $a^2 + 5b^2 + 4c^2 - 4ab - 4bc = 0$. Can a, b, c be the lengths of the sides of a triangle? Justify your answer.

Solution 1. The given condition is equivalent to

$$(a - 2b)^2 + (b - 2c)^2 = 0.$$

Since a, b, c are real numbers, this gives $b = 2c, a = 2b = 4c$. Note that $b + c = 3c < 4c = a$ holds. So by the triangle inequality, it follows that a, b, c cannot be the lengths of the sides of a triangle. ■

Exercise 2 (RMO 2014e P2,). The roots of the equation

$$x^3 - 3ax^2 + bx + 18c = 0$$

form a non-constant arithmetic progression and the roots of the equation

$$x^3 + bx^2 + x - c^3 = 0$$

form a non-constant geometric progression. Given that a, b, c are real numbers, find all positive integral values a and b.

Solution 2. Let a, b, c are real numbers such that the given conditions hold. Let $d \neq 0$ denote the common difference of the non-constant arithmetic progression formed by the roots of $x^3 - 3ax^2 + bx + 18c = 0$, and let r denote the common ratio of the non-constant geometric progression formed by the roots of $x^3 + bx^2 + x - c^3 = 0$. Note that the terms of the arithmetic progression are $a - d, a, a + d$, and the terms of the geometric progression are $\frac{c}{r}, c, cr$. It follows that

$$a(a - d) + a(a + d) + (a^2 - d^2) = b, \quad a(a^2 - d^2) = -18c,$$

which gives

$$d^2 = 3a^2 - b, \quad a(a^2 - d^2) = -18c,$$

and

$$\frac{c}{r} + c + cr = -b, \frac{c^2}{r} + c^2 + c^2 r = 1.$$

Eliminating d, we obtain

$$a(4a^2 - b) = -18c,$$

and eliminating r, we obtain

$$bc = -1.$$

Eliminating c yields

$$ab(b - 2a^2) = 18. \tag{1}$$

Assume that a, b are positive integers. Since a, b are positive, it follows that $b - 2a^2$ is also positive. Note that $b, b - 2a^2$ are of the same parity. Since 2 divides 18 and 4 does not divide 18, we obtain $a = 2$, and hence

$$b(b - 8) = 9.$$

Since b is positive, it gives $b = 9$.

Note that for $(a, b) = (2, 9)$, and for $c = -\frac{1}{9}$, the roots of the polynomial $x^3 - 3ax^2 + bx + 18c$ are $2 \mp \sqrt{3}, 2, 2 \pm \sqrt{3}$, which form a non-constant arithmetic progression, and the roots of the polynomial $x^3 + bx^2 + x - c^3$ are $c/r, c, cr$ with $r = 40 \pm \sqrt{40^2 - 1}$, which form a non-constant geometric progression.

This proves that the required solution for (a, b) is precisely $(2, 9)$. ■

Exercise 3 (RMO 2014e P4,). A person moves in the x-y plane moving along points with integer co-ordinates x and y only. When she is at a point (x, y), she takes a step based on the following rules:

(a) if $x + y$ is even she moves to either $(x + 1, y)$ or $(x + 1, y + 1)$,

(b) if $x + y$ is odd she moves to either $(x, y + 1)$ or $(x + 1, y + 1)$.

How many distinct paths can she take to go from $(0, 0)$ to $(8, 8)$ given that she took exactly three steps to the right ((x, y) to $(x + 1, y)$)?

Solution 3. If R, U, D denote the number of steps taken to the right, upwards and along the diagonal, then the coordinate of the final point is $(R + D, U + D)$. Since R is equal to 3, it follows that D is equal to 5 and U is equal to 3. Note that between two consecutive steps to the right, there is an odd number of upward steps. So between two consecutive steps to the right, there is exactly one upward step. Thus the moves to the right and the upward moves form the sequence $RURURU$ or $URURUR$. Since the sum of the coordinates of the initial point is even and a diagonal move does change the parity of this sum, a move to the right is taken before the first upward move. So the moves to the right and the upward moves form the sequence $RURURU$. Hence given a path

from $(0,0)$ to $(8,8)$, the five diagonal moves are placed in the blanks below (a blank space might contain no diagonal move or more than one diagonal move). Conversely, any filling of the following blanks with five D's give us a path from $(0,0)$ to $(8,8)$.

$$-R-U-R-U-R-U-$$

So the number of distinct paths is equal to the number of solutions of the equation

$$x_1+x_2+\cdots+x_7=5$$

in nonnegative integers, which is equal to $\binom{5+7-1}{6}=\binom{11}{6}$. ■

Exercise 4 (RMO 2014e P5,). Let a,b,c be positive real numbers such that

$$\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\le 1. \tag{2}$$

Prove that

$$(1+a^2)(1+b^2)(1+c^2)\ge 125.$$

When does the equality hold?

Solution 4. The given inequality implies

$$\frac{a}{1+a}\ge\frac{1}{1+b}+\frac{1}{1+c}\ge\frac{2}{\sqrt{(1+b)(1+c)}},$$

where the second inequality holds if and only if $b=c$. Similar lower bounds for $\frac{b}{1+b},\frac{c}{1+c}$ can be obtained, and multiplying them yields $abc\ge 8$, where equality holds if and only if $a=b=c=2$. Note that

$$1+a^2=1+\frac{a^2}{4}+\frac{a^2}{4}+\frac{a^2}{4}+\frac{a^2}{4}\ge 5\left(\frac{a^8}{2^8}\right)^{\frac{1}{5}}$$

holds, where equality occurs if and only if $a=2$. We can find similar lower bounds for $1+b^2,1+c^2$. Multiplying them, we get

$$(1+a^2)(1+b^2)(1+c^2)\ge 5^3\left(\frac{a^8}{2^8}\frac{b^8}{2^8}\frac{c^8}{2^8}\right)^{\frac{1}{5}}=5^3\left(\frac{abc}{8}\right)^{\frac{8}{5}}\ge 125,$$

where equality occurs if and only if $a=b=c=2$. ■