RMO 2014d

CRMO 4

Exercise 1 (RMO 2014d P2,). If x and y are positive real numbers, prove that

$$4x^4 + 4y^3 + 5x^2 + y + 1 \geq 12xy.$$

Solution 1. Note that

$$\begin{aligned} 4x^4 + 4y^3 + 5x^2 + y + 1 &\geq (4x^4 + 1) + 5x^2 + (4y^3 + y) \\ &= (2x^2 - 1)^2 + 9x^2 + y(2y - 1)^2 + 4y^2 \\ &\geq 9x^2 + 4y^2 \\ &= (3x - 2y)^2 + 12xy \\ &\geq 12xy. \end{aligned}$$

■

Exercise 2 (RMO 2014d P3,). Determine all pairs $m > n$ of positive integers such that

$$1 = \gcd(n + 1, m + 1) = \gcd(n + 2, m + 2) = \cdots = \gcd(m, 2m - n).$$

Solution 2. Let $m > n$ be positive integers satisfying the given condition. It follows that

$$1 = \gcd(n + 1, m - n) = \gcd(n + 2, m - n) = \cdots = \gcd(m, m - n).$$

Note that the positive integer $m-n$ is relatively prime to the $m-n$ consecutive integers

$$n + 1, n + 2, \ldots, m.$$

This implies that $m - n = 1$.

Note that if $m - n = 1$ holds for two positive integers, then

$$1 = \gcd(n + 1, m - n) = \gcd(n + 2, m - n) = \cdots = \gcd(m, m - n)$$

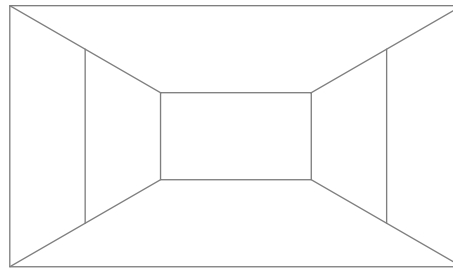


Figure 1: India RMO 2014d P6, Exercise 3

holds, and which yields

$$1 = \gcd(n+1, m+1) = \gcd(n+2, m+2) = \cdots = \gcd(m, 2m-n).$$

This proves that the pairs (m, n) satisfying the given condition are precisely of the pairs of the form $(n+1, n)$ as n ranges over the positive integers. ■

Exercise 3 (RMO 2014d P6,). In Fig. 1, can the numbers $1, 2, 3, 4, \ldots, 18$ be placed, one on each line segment, such that the sum of the numbers on the three line segments meeting at each point is divisible by 3?

Summary — Since there are 18 line segments, it follows that if the integers $0, 1, 2$ can be put on the segments, using each of them exactly six times, such that 3 divides the sum of the integers on the segments meeting at any given point, then it would be possible to place $1, 2, \ldots, 18$ satisfying the required condition.

Solution 3. Note that if the integers $0, 1, 2$ can be put on the segments, using each of them exactly six times, such that 3 divides the sum of the integers on the segments meeting at any given point, then it would be possible to place $1, 2, \ldots, 18$ satisfying the required condition (by replacing the 0's (resp. 1's, 2's) by the six integers among $1, 2, \ldots, 18$ which are congruent to 0 (resp. $1, 2$) modulo 3, and such a replacement can be carried out since there are six elements among $1, 2, \ldots, 18$ congruent to $i \bmod 3$ for any $i \in \{0, 1, 2\}$). We now show that such an arrangement of $0, 1, 2$ exists. First, put 0's on all the vertical segments as in Fig. 2b, and then put 1's on the 'diagonal' segments as shown in Fig. 2c. This forces to put 2's on the 'diagonal' segments as shown in Figure 2d, which in turn, forces to write 1's and 2's on the horizontal segments as in Figure 2e. Note that the sum of the numbers (as in Figure 2e) on the three line segments meeting at each point is divisible by 3. So this gives an arrangement of $0, 1, 2$ satisfying the desired properties, then $1, 2, \ldots, 18$ can be arranged satisfying the given conditions (as described above). ■

Exercise 3 leads to the following question.

The content posted here and at this blog by Evan Chen are quite useful.

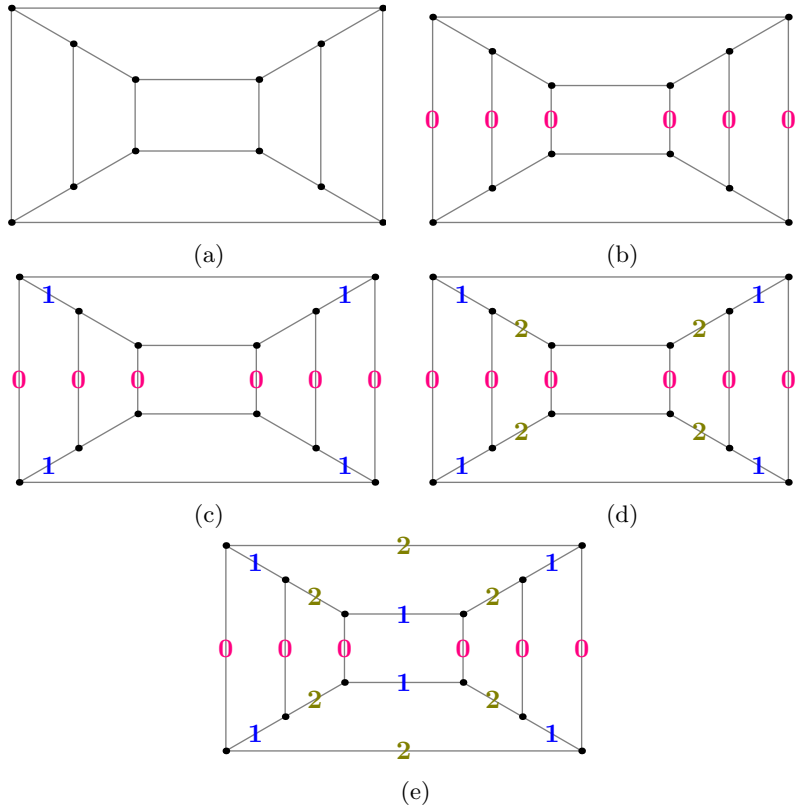


Figure 2: India RMO 2014, Exercise 3

Question 4. Under which conditions, does a k-regular graph admit an edge coloring by the k-th roots of unity such that the sum of the colors incident at any vertex equals to zero?