RMO 2014c

CRMO 3

Exercise 1 (RMO 2014c P2,). Find all real x, y such that

$$x^2 + 2y^2 + \frac{1}{2} \leq x(2y+1).$$

Solution 1. Let x, y be reals satisfying the above inequality. Note that

$$\begin{aligned} x^2 + 2y^2 + \frac{1}{2} - x(2y+1) &= x^2 + 2y^2 + \frac{1}{2} - 2xy - x \\ &= 2(y^2 - xy) + x^2 - x + \frac{1}{2} \\ &= 2\left(y - \frac{x}{2}\right)^2 + \frac{x^2}{2} - x + \frac{1}{2} \\ &= 2\left(y - \frac{x}{2}\right)^2 + \frac{1}{2}(x-1)^2 \\ &\geq 0. \end{aligned}$$

For real x, y, the given inequality is equivalent to $x - 1 = 0, y - \frac{x}{2} = 0$, which holds if and only if (x, y) is equal to $(1, \frac{1}{2})$. ■

Exercise 2 (RMO 2014c P3,). Prove that for any natural number $n < 2310$, $n(2310 - n)$ is not divisible by 2310.

Solution 2. Suppose $n < 2310$ is a positive integer such that $n(2310 - n)$ is equal to $2310k$ for some positive integer k. It follows that there are positive integers a, b such that a divides n, b divides $2310 - n$ and $ab = 2310$. Note that b also divides n. This shows that n is divisible by the least common multiple of the integers a and b, that is, by $ab/\gcd(a, b)$. Note that

$$2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$$

holds, and hence 2310 is a square-free integer. Since ab is equal to 2310, it follows that $\gcd(a, b) = 1$. This shows that ab divides n, that is, 2310 divides n, which is impossible. This completes the proof. ■

Exercise 3 (RMO 2014c P4,). Find all positive reals x, y, z such that

$$2x - 2y + \frac{1}{z} = \frac{1}{2014},$$
$$2y - 2z + \frac{1}{x} = \frac{1}{2014},$$
$$2z - 2x + \frac{1}{y} = \frac{1}{2014}.$$

Solution 3. Let x, y, z be positive real numbers satisfying the given equations. Adding these equations yields

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{3}{2014}.$$

Multiplying these equations by z, x, y respectively and then by adding them, we obtain

$$3 = \frac{x}{2014} + \frac{y}{2014} + \frac{z}{2014}.$$

This implies that

$$\frac{x+y+z}{3} = \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$$

holds. Since x, y, z are positive real numbers, by AM-HM inequality, it follows that x, y, z are equal. Hence x, y, z are equal to 2014.

Note that $(x, y, z) = (2014, 2014, 2014)$ satisfies the given equations. It follows that $(2014, 2014, 2014)$ is the only solution of the given system of equations over the positive reals. ■

Exercise 4 (RMO 2014c P6,). For any natural number, let $S(n)$ denote sum of digits of n. Find the number of 3 digit numbers for which $S(S(n)) = 2$.

Solution 4. Let n be a 3-digit natural number such that $S(S(n)) = 2$. Note that $S(n) \leq 27$. Using $S(n) \leq 27$ and $S(S(n)) = 2$, it follows that $S(n)$ is equal to one of $2, 11, 20$. Noting that $S(n)$ is congruent to 2 mod 9, and using $n \equiv S(n) \bmod 9$, we obtain $n \equiv 2 \bmod 9$. Conversely, note that if m is a 3-digit number and $m \equiv 2 \bmod 9$, then $S(m) \equiv 2 \bmod 9$ holds, and using $S(m) \leq 27$, it follows that $S(m)$ is equal to one of $2, 11, 20$. We conclude that the 3-digit numbers n satisfying $S(S(n)) = 2$ are precisely the 3-digit numbers which are congruent to 2 modulo 9, or equivalently, n is equal to one of

$$101, 110, \ldots, 992.$$

Hence there are

$$1 + \frac{1}{9}(992 - 101) = 100$$

three-digit numbers n for which $S(S(n)) = 2$. ■

The content posted here and at this blog by Evan Chen are quite useful.