RMO 2014b

CRMO 2

Exercise 1 (RMO 2014b P2,). Let x, y, z be positive real numbers. Prove that

$$\frac{y^2+z^2}{x}+\frac{z^2+x^2}{y}+\frac{x^2+y^2}{z}\geq 2(x+y+z).$$

Solution 1. Note that

$$\frac{y^2+z^2}{x}+\frac{z^2+x^2}{y}+\frac{x^2+y^2}{z}=\left(\frac{y^2}{x}+\frac{z^2}{y}+\frac{x^2}{z}\right)+\left(\frac{z^2}{x}+\frac{x^2}{y}+\frac{y^2}{z}\right).$$

holds. Since (x^2, y^2, z^2) and $(\frac{1}{x}, \frac{1}{y}, \frac{1}{z})$ are sorted oppositely, by the rearrangement inequality, we obtain

$$\left(\frac{y^2}{x}+\frac{z^2}{y}+\frac{x^2}{z}\right)\geq x+y+z, \quad \left(\frac{z^2}{x}+\frac{x^2}{y}+\frac{y^2}{z}\right)\geq x+y+z.$$

Adding these inequalities, the required inequality follows. ■

> **Remark.** Note that the last two inequalities also follow from the Cauchy–Schwarz inequality.

Exercise 2 (RMO 2014b P3,). Find all pairs of (x, y) of positive integers such that $2x + 7y$ divides $7x + 2y$.

Solution 2. Let (x, y) be a pair of positive integers such that $2x + 7y$ divides $7x+2y$. Let d denote the positive integer such that $7x+2y$ is equal to $(2x+7y)d$. This gives

$$7(x - dy) = 2(dx - y).$$

Since the integers $2, 7$ are relatively prime, it follows that 2 divides $x - dy$ and 7 divides $dx - y$. This shows that

$$\frac{x-dy}{2}=\frac{dx-y}{2}=k$$

holds for some integer k. This shows that

$$x - dy = 2k, \quad dx - y = 7k,$$

which gives

$$(d^2-1)x = (7d-2)k, \quad (d^2-1)y = (7-2d)k.$$

Since y is positive, we obtain $d \leq 3$. Note that if $d = 1$, then $k = 0$ and hence $x = y$. If $d = 2$, then (x, y) is equal to $(4k, k)$, and if $d = 3$, then (x, y) is equal to $(\frac{19}{8}k, \frac{1}{8}k)$, in which case k is a multiple of 8. So (x, y) is an element of the set

$$\bigcup_{r=1}^{\infty} \{(r, r), (4r, r), (19r, r)\}.$$

Also note that for any element (a, b) of this set, the integer $2a + 7b$ divides $7a + 2b$. This proves that the required pairs form the above set. ■

Exercise 3 (RMO 2014b P4,). For any positive integer $n > 1$, let $P(n)$ denote the largest prime not exceeding n. Let $N(n)$ denote the next prime larger than $P(n)$. (For example $P(10) = 7$ and $N(10) = 11$, while $P(11) = 11$ and $N(11) = 13$.) If $n + 1$ is a prime number, prove that the value of the sum

$$\frac{1}{P(2)N(2)} + \frac{1}{P(3)N(3)} + \frac{1}{P(4)N(4)} + \cdots + \frac{1}{P(n)N(n)} = \frac{n-1}{2n+2}.$$

Solution 3. Let p_n denote the n-th prime for $n \geq 1$. Note that for any integer m with $p_n \leq m \leq p_{n+1} - 1$, the integer $P(m)N(m)$ is equal to $p_n p_{n+1}$. This gives

$$\sum_{m=p_n}^{p_{n+1}-1} \frac{1}{P(m)N(m)} = \frac{p_{n+1} - p_n}{p_n p_{n+1}} = \frac{1}{p_n} - \frac{1}{p_{n+1}}.$$

Let k denote the positive integer such that $n + 1$ is equal to p_k. Note that

$$\begin{aligned}
&\frac{1}{P(2)N(2)} + \frac{1}{P(3)N(3)} + \frac{1}{P(4)N(4)} + \cdots + \frac{1}{P(n)N(n)} \\
&= \sum_{m=p_1}^{p_2-1} \frac{1}{P(m)N(m)} + \sum_{m=p_2}^{p_3-1} \frac{1}{P(m)N(m)} + \cdots + \sum_{m=p_{k-1}}^{p_k-1} \frac{1}{P(m)N(m)} \\
&= \left(\frac{1}{p_1} - \frac{1}{p_2}\right) + \left(\frac{1}{p_2} - \frac{1}{p_3}\right) + \cdots + \left(\frac{1}{p_{k-1}} - \frac{1}{p_k}\right) \\
&= \frac{1}{p_1} - \frac{1}{p_k} \\
&= \frac{1}{2} - \frac{1}{n+1}
\end{aligned}$$

The content posted here and at this blog by Evan Chen are quite useful.

$$= \frac{n-1}{2n+2}.$$

■

Exercise 4 (RMO 2014b P6,). [**Eng98**, Problem 28, p. 29, p. 35] Let n be an odd positive integer and suppose that each square of an $n \times n$ grid is arbitrarily filled with either by 1 or by -1. Let r_j and c_k denote the product of all numbers in j-th row and k-th column respectively, $1 \leq j, k \leq n$. Prove that

$$\sum_{j=1}^{n} r_j + \sum_{k=1}^{n} c_k \neq 0.$$

Walkthrough —

(a) What would happen to the sum if the sign of one of the entries is changed? Does the sum change modulo 4?

(b) What would happen when all the entries are changed to 1 by changing the signs of the negative entries?

Solution 4. Denote the sum $\sum_{j=1}^{n} r_j + \sum_{k=1}^{n} c_k$ by S. If all entries of the grid are equal to 1, then S is equal to $2n$. Note that if the sign on an entry of the grid is changed, then S does not change modulo 4. So the integer S is congruent to $2n$ modulo 4. Since n is odd, the integer S is nonzero. For an alternate solution, see [**Eng98**, Problem 29, Chapter 2, p. 28]. ■

References

[**Eng98**] Arthur Engel. *Problem-solving strategies*. Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. ISBN: 0-387-98219-1 (cited p. 3)