RMO 2014a

CRMO 1

Exercise 1 (RMO 2014a P2,). Let $a_1, a_2, \ldots, a_{2n}$ be an arithmetic progression of positive real numbers with common difference d. Let

$$\sum_{i=1}^{n} a_{2i-1}^2 = x, \quad \sum_{i=1}^{n} a_{2i}^2 = y, \quad a_n + a_{n+1} = z.$$

Express d in terms of x, y, z, n.

Solution 1. Note that

$$\begin{aligned} y - x &= (a_2^2 - a_1^2) + (a_4^2 - a_3)^2 + \cdots + (a_{2n}^2 - a_{2n-1}^2) \\ &= d(a_1 + a_2 + a_3 + a_4 + \cdots + a_{2n-1} + a_{2n}). \end{aligned}$$

Note that $a_i + a_{2n-i}$ is equal to $a_n + a_{n+1}$ for any $0 < i < 2n$. Indeed,

$$\begin{aligned} a_i + a_{2n-i} &= a_1 + (i-1)d + a_1 + (2n-i-1)d \\ &= 2a_1 + (2n-2)d \end{aligned}$$

is independent of i. This shows that

$$y - x = dn(a_n + a_{n+1}) = dnz.$$

which is equal to dnz. Since $a_1, a_2, \ldots, a_{2n}$ is an arithmetic progression of positive real numbers, it follows that $z = a_n + a_{n+1}$ is nonzero, and consequetly,

$$d = \frac{y - x}{nz}.$$

■

Exercise 2 (RMO 2014a P3, , cf. Tournament of Towns, Fall 1988, Training). [Tao06, Problem 2.2] [AE11, Problem 3.81] Suppose for some positive integers r and s, the integer 2^r is obtained by permuting the digits of 2^s in decimal expansion and $2^r, 2^s$ have same number of digits. Prove that $r = s$.

Walkthrough —

(a) On the contrary, assume that $r \neq s$. Without loss of generality, we may assume that $r > s$ holds.

(b) Show that $2^{r-s} - 1$ is divisible by 9.

(c) Prove that 6 divides $r - s$.

(d) Conclude that $2^r \geq 64 \cdot 2^s$ holds, contradicting the fact that 2^r and 2^s have the same number of digits.

Solution 2. Since a positive integer is congruent to the sum of its digits modulo 9, it follows that 2^r and 2^s are congruent modulo 9.

Let us consider the case that $r < s$. Note that 9 divides $2^{s-r} - 1$. Since the order of 2 modulo 9 is equal to 6, it follows that 6 divides $s - r$, and hence $2^s \geq 64 \cdot 2^r$, which is impossible. This shows that $r \geq s$ holds. Similarly, it also follows that $s \geq r$ holds. This proves that $s = r$, as required. ■

Exercise 3 (RMO 2014a P4,). Is it possible to write the numbers $17, 18, 19, \ldots, 32$ in a 4×4 grid of unit squares with one number in each square such that if the grid is divided into four 2×2 subgrids of unit squares, then the product of numbers in each of the subgrids divisible by 16?

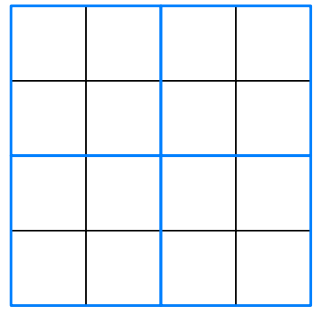


Figure 1: RMO 2014a P4, Exercise 3

Walkthrough —

(a) Note that the product of the entries in some subgrid is divisible by 32.

(b) Conclude that the product of all the 16 entries is divisible by $16 \times 16 \times 16 \times 32$.

(c) Is the product of the integers $17, 18, \ldots, 32$ divisible by $16 \times 16 \times 16 \times 32$?

Solution 3. The highest exponents of 2 dividing 32! and 16! are given by

$$v_2(32!) = 16 + 8 + 4 + 2 + 1, \quad v_2(16!) = 8 + 4 + 2 + 1.$$

So the highest power of 2 dividing the product of the integers $17, 18, 19, \ldots, 32$ is 2^{16}. Now if it were possible to write these numbers in a 4×4 grid in the above-mentioned manner, then the product of the numbers in each of the 2×2 subgrids with blue boundary (see Fig. 1) would be divisible by 2^4. Note that one such subgrid would contain 32, which implies that the product of $17, 18, \ldots, 32$ is divisible by $2^4 \cdot 2^4 \cdot 2^4 \cdot 32 = 2^{17}$, which is impossible. Hence it is not possible to write the integers $17, 18, \ldots, 32$ in a 4×4 grid satisfying the given conditions. ■

Exercise 4 (RMO 2014a P6,). Let $x_1, x_2, x_3, \ldots, x_{2014}$ be positive real numbers such that $\sum_{j=1}^{2014} x_j = 1$. Determine with proof the smallest constant K such that

$$K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1. \tag{1}$$

Solution 4. For any positive reals $x_1, \ldots, x_{2014}$ satisfying $\sum_{j=1}^{2014} x_j = 1$, using the Cauchy–Schwarz inequality, we obtain

$$\sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq \frac{(x_1 + \cdots + x_{2014})^2}{2014 - (x_1 + \cdots + x_{2014})} = \frac{1}{2013},$$

which gives

$$2013 \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1.$$

This shows that the inequality (**??**) holds for $K = 2013$.

For $x_1 = x_2 = \cdots = x_{2014} = \frac{1}{2014}$, note that

$$\sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} = \frac{1}{2013}$$

holds, which shows that for any K satisfying the inequality (**??**), the bound $K \geq 2013$ holds.

This proves that the smallest constant K satisfying the inequality (**??**) is equal to 2013. ■

References

[**AE11**] Titu Andreescu and Bogdan Enescu. *Mathematical Olympiad treasures*. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. ISBN: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 1)

[**Tao06**] Terence Tao. *Solving mathematical problems*. A personal perspective. Oxford University Press, Oxford, 2006, pp. xii+103. ISBN: 978-0-19-920560-8; 0-19-920560-4 (cited p. 1)