RMO 2013e

Mumbai

Exercise 1 (RMO 2013e P2,). Find all triples (p, q, r) of primes such that $pq = r + 1$ and $2(p^2 + q^2) = r^2 + 1$.

Solution 1. Let p, q, r be primes satisfying the above conditions. Since 2 divides $r^2 + 1$, it follows that r is odd. This shows that pq is even, and hence at least one of p, q is even.

Let us consider the case that $p = 2$. We obtain $q = \frac{r+1}{2}$. This yields

$$2\left(2^2 + \frac{(r+1)^2}{2^2}\right) = r^2 + 1,$$

which implies

$$r^2 - 2r - 15 = 0,$$

which shows that $r = 5$, and hence, (p, q, r) is equal to $(2, 3, 5)$. Since the given equations are symmetric in p, q, it follows that if $q = 2$, then (p, q, r) is equal to $(3, 2, 5)$. Note that the triples $(2, 3, 5)$ and $(3, 2, 5)$ satisfy the given equations.

We conclude that $(2, 3, 5), (3, 2, 5)$ are precisely all the triples of primes satisfying the given conditions. ■

Exercise 2 (RMO 2013e P3,). A finite non-empty set of integers is called 3-*good* if the sum of its elements is divisible by 3. Find the number of 3-good subsets of $\{0, 1, 2, \ldots, 9\}$.

Solution 2. For any subset of S of $\{0, 1, 2, \ldots, 9\}$, denote by s_0 (resp. s_1, s_2) the number of elements S which are congruent to 0 (resp. 1, 2) modulo 3. Note that a nonempty subset S of $\{0, 1, 2, \ldots, 9\}$ is 3-good if and only if $0 \cdot s_0 + 1 \cdot s_1 + 2 \cdot s_2 \equiv 0 \bmod 3$, that is, $s_1 \equiv s_2 \bmod 3$. The elements of $\{0, 1, 2, \ldots, 9\}$ congruent to 0 (resp. 1, 2) modulo 3 are $0, 3, 6, 9$ (resp. $1, 4, 7$, and $2, 5, 8$). So the number of 3-good subsets of $\{0, 1, 2, \ldots, 9\}$ is equal to

$$\begin{aligned}&\binom{3}{0}^2(2^4 - 1) + \binom{3}{1}^2 2^4 + \binom{3}{2}^2 2^4 + \binom{3}{3}^2 2^4 + \binom{3}{0}\binom{3}{3}2^4 + \binom{3}{3}\binom{3}{0}2^4 \\ &= 2^4(1 + 9 + 9 + 1 + 1 + 1) - 1\end{aligned}$$

$= 351.$

■

Exercise 3 (RMO 2013e P5,). Let a_1, b_1, c_1 be natural numbers. We define

$$a_2 = \gcd(b_1, c_1), \quad b_2 = \gcd(c_1, a_1), \quad c_2 = \gcd(a_1, b_1)$$

and

$$a_3 = \operatorname{lcm}(b_2, c_2), \quad b_3 = \operatorname{lcm}(c_2, a_2), \quad c_3 = \operatorname{lcm}(a_2, b_2).$$

Show that $\gcd(b_3, c_3) = a_2$.

Solution 3. It suffices to show that the highest power of p dividing $\gcd(b_3, c_3)$ and a_2 are equal for any prime p. Fix a prime number p. Let p^x, p^y, p^z denote the highest powers of p dividing a_1, b_1, c_1 respectively. So the highest exponents of p dividing a_2, b_2, c_2 are

$$\min\{y, z\}, \min\{z, x\}, \min\{x, y\}$$

respectively. Thus the highest exponents of p dividing b_3, c_3 are

$$\max\{\min\{x, y\}, \min\{y, z\}\}, \max\{\min\{y, z\}, \min\{z, x\}\}$$

respectively. So it remains to show that

$$\min\{\max\{\min\{x, y\}, \min\{y, z\}\}, \max\{\min\{y, z\}, \min\{z, x\}\}\} = \min\{y, z\} \tag{1}$$

holds. Note that $\min\{x, y\}, \min\{y, z\}$ are less than or equal to y, which gives

$$\max\{\min\{x, y\}, \min\{y, z\}\} \leq y.$$

Similarly, it follows that

$$\max\{\min\{y, z\}, \min\{z, x\}\} \leq z.$$

This gives that

$$\min\{\max\{\min\{x, y\}, \min\{y, z\}\}, \max\{\min\{y, z\}, \min\{z, x\}\}\} \leq \min\{y, z\}.$$

Note also that

$$\max\{\min\{x, y\}, \min\{y, z\}\} \geq \min\{y, z\},$$
$$\max\{\min\{y, z\}, \min\{z, x\}\} \geq \min\{y, z\}$$

holds, which yields

$$\min\{\max\{\min\{x, y\}, \min\{y, z\}\}, \max\{\min\{y, z\}, \min\{z, x\}\}\} \geq \min\{y, z\}.$$

This proves that Eq. (1) holds. ■

The content posted here and at this blog by Evan Chen are quite useful.

Exercise 4 (RMO 2013e P6,). Let $P(x) = x^3 + ax^2 + b$ and $Q(x) = x^3 + bx + a$, where a and b are nonzero real numbers. Suppose that the roots of the equation $P(x) = 0$ are the reciprocals of the roots of the equation $Q(x) = 0$. Prove that a and b are integers. Find the greatest common divisor of $P(2013! + 1)$ and $Q(2013! + 1)$.

Solution 4. The reciprocals of the roots of the polynomial $Q(x)$ are the roots of polynomial

$$R(x) := x^3 P\left(\frac{1}{x}\right) = ax^3 + bx^2 + 1.$$

The given condition implies that $R(x) = aP(x)$, that is, $ab = 1, a^2 = b$. This gives $a^3 = 1$. Since a is real, it follows that $a = 1$ and hence $b = 1$. This proves that a, b are integers.

Note that for any integer n,

$$\begin{aligned}
\gcd(P(n), Q(n)) &= \gcd(n^3 + n^2 + 1, n^3 + n + 1) \\
&= \gcd(n^2 - n, n^3 + n + 1) \\
&= \gcd(n^2 - n, n^3 + n + 1 - (n^2 - n)(n + 1)) \\
&= \gcd(n^2 - n, 2n + 1) \\
&= \gcd(n(n - 1), 2n + 1) \\
&= \gcd(n - 1, 2n + 1) \\
&= \gcd(n - 1, 2(n - 1) + 3) \\
&= \gcd(n - 1, 3).
\end{aligned}$$

For $n = 2013! + 1$, note that $\gcd(n - 1, 3) = 1$, and hence the greatest common divisor of $P(2013! + 1)$ and $Q(2013! + 1)$ is equal to 3. ■