RMO 2013d

CRMO 4

Exercise 1 (RMO 2013d P2,). Determine the smallest prime that does not divide any five-digit number whose digits are in a strictly increasing order.

Solution 1. Note that 2 divides 12346, 3 and 5 divide 12345, and 7 divides 12348. Hence, the smallest prime satisfying the required condition is at least as large as 11.

Let $(abcde)_{10}$ be a five-digit number in base 10 with $1 \leq a < b < c < d < e \leq 9$. Note that

$$a - b + c - d + e = (a - b) + (c - d) + e \leq e \leq 9$$

holds, and

$$a - b + c - d + e = a + (c - b) + (e - d) \geq a \geq 1$$

holds. This shows that 11 does not divide $a - b + c - d + e$, or equivalently 11 does not divide the integer $(abcde)_{10}$. Since 11 does not divide any five-digit number whose digits are in a strictly increasing order, it follows that the smallest prime satisfying the required condition is at most 11.

We conclude that 11 is the smallest prime satisfying the required condition. ■

Exercise 2 (RMO 2013d P3,). Given real numbers $a, b, c, d, e > 1$. Prove that

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq 20.$$

Solution 2. Since $a-1, b-1, c-1, d-1, e-1$ are all positive, by applying the Cauchy-Schwarz inequality, we get

$$\frac{a^2}{c-1} + \frac{b^2}{d-1} + \frac{c^2}{e-1} + \frac{d^2}{a-1} + \frac{e^2}{b-1} \geq \frac{(a+b+c+d+e)^2}{a+b+c+d+e-5}.$$

It suffices to show that

$$\frac{(a+b+c+d+e)^2}{a+b+c+d+e-5} \geq 20$$

holds. Since $a+b+c+d+e-5$ is positive, it is enough to prove that

$$(a+b+c+d+e)^2-20(a+b+c+d+e)+100\geq 0,$$

which holds

$$(a+b+c+d+e)^2-20(a+b+c+d+e)+100=(a+b+c+d+e-10)^2.$$

This proves the result. ■

Exercise 3 (RMO 2013d P4,). Let x be a nonzero real number such that $x^4+\frac{1}{x^4}$ and $x^5+\frac{1}{x^5}$ are both rational numbers. Prove that $x+\frac{1}{x}$ is a rational number.

Summary — Consider the difference

$$\left(x^5+\frac{1}{x^5}\right)-\left(x^4+\frac{1}{x^4}\right)\left(x+\frac{1}{x}\right).$$

Solution 3. For a positive integer n, put $y_n=x^n+\frac{1}{x^n}$. Note that

$$y_5=y_1y_4-y_3=y_1y_4-y_1(y_2-1)=y_1(y_4-y_2+1),$$

and y_4-y_2+1 is nonzero, otherwise, we would get $y_2^2-y_2-1=0$, that is, $y_2=\frac{1\pm\sqrt{5}}{2}$, which shows that $y_4=\frac{3\pm\sqrt{5}}{2}$, which contradicts that y_4 is rational. To show that y_1 is rational, it suffices to show that y_2 is rational. Observe that

$$y_{10}=y_2y_8-y_6=y_2y_8-y_2y_4+y_2=y_2(y_8-y_4+1),$$

and $y_8-y_4+1\neq 0$ (otherwise, we would obtain $y_4^2-y_4-1=0$, which contradicts the rationality of y_4). Since y_{10},y_8,y_4 are rational, it follows that y_2 is rational. Using

$$y_5=y_1(y_4-y_2+1),$$

and that y_4-y_2+1 is nonzero, we conclude that y_1 is rational. ■

Exercise 4 (RMO 2013d P6,). Suppose that the vertices of a regular polygon of 20 sides are coloured with three colors red, blue and green, such that there are exactly three red vertices. Prove that there are three vertices A,B,C of the polygon having the same colour such that triangle ABC is isosceles.

Walkthrough — Decompose the set of vertices of the 20-gon using the vertices of the pentagons as in Fig. 1. Apply the pigeonhole principle. It would be useful to note that any three vertices of a regular pentagon form the vertices of an isosceles triangle.

The content posted here and at this blog by Evan Chen are quite useful.

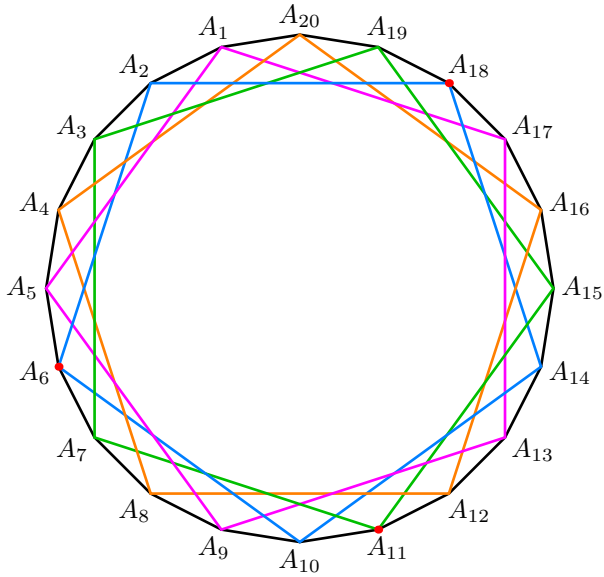


Figure 1: RMO 2013d P6, Exercise 4

Solution 4. Since there are exactly three red vertices and any of the remaining 17 vertices are blue or green, it follows that at least 9 of these 17 vertices are of the same color, say blue. Note that the set of vertices of a regular 20-gon can be written as the union of the four pairwise disjoint sets, each of them consisting of the vertices of a regular pentagon (as in Fig. 1). Since there are nine blue vertices, by the pigeonhole principle, at least one of these four sets contains three blue points. Since any three points on a regular pentagon form the vertices of an isosceles triangle, the statement follows. ■