RMO 2013c

CRMO 3

Exercise 1 (RMO 2013c P1,). Find the number of eight-digit numbers the sum of whose digits is 4.

Solution 1. Note that number of eight-digit numbers with sum of digits equal to 4 is same as the number of solutions of

$$x_1 + x_2 + \cdots + x_8 = 4$$

with $x_1 \geq 1$ and $0 \leq x_1, \ldots, x_8 \leq 9$. It is also equal to the number of solutions of

$$y_1 + y_2 + \cdots + y_8 = 3$$

in nonnegative integers, which is $\binom{3+8-1}{8-1} = \binom{10}{7} = 120$. ■

Exercise 2 (RMO 2013c P2,). Find all 4-tuples (a, b, c, d) of natural numbers with $a \leq b \leq c$ and $a! + b! + c! = 3^d$.

Solution 2. Let a, b, c, d be natural numbers satisfying the given conditions. Note that $d \geq 1$. If $a \geq 2$, then $3^d = a! + b! + c!$ would be even, which is impossible. So a is equal to 1, which gives $b! + c! = 3^d - 1$. This implies that $b \leq 2$. If $b = 1$, then $c! = 3^d - 2$, which shows $c \leq 2$, and hence $c = 1, d = 1$. Suppose b is equal to 2. Then we obtain $c! = 3^d - 3$, which gives $d \geq 2$. Since 9 does not divide $3^d - 3$, it follows that $c \leq 5$. This implies that (c, d) is equal to $(3, 2)$ or $(4, 3)$. This gives that (a, b, c, d) is equal to one of

$$(1, 1, 1, 1), (1, 2, 3, 2), (1, 2, 4, 3).$$

Note that the above tuples also satisfy the given conditions. This shows that the tuples are precisely the 4-tuples of natural numbers satisfying the given conditions. ■

Exercise 3 (RMO 2013c P4,). A polynomial is called a *Fermat polynomial* if it can be written as the sum of squares of two polynomials with integer coefficients. Suppose that $f(x)$ is a Fermat polynomial such that $f(0) = 1000$. Prove that $f(x) + 2x$ is not a Fermat polynomial.

Summary — Assume that $f(x)+2x$ is a Fermat polynomial. Write each of the polynomials $f(x), f(x)+2x$ as a sum of the squares of two polynomials with integer coefficients. Reducing modulo 4, conclude that the constant terms of these four polynomials are even. Next, reduce modulo x^2, and compare the coefficients of x to obtain a contradiction.

Solution 3. On the contrary, let us assume that $f(x)+2x$ is a Fermat polynomial.

Since $f(x)$ is a Fermat polynomial, it follows that there exist polynomials $P(x)$ and $Q(x)$ with integer coefficients such that $f(x)=P(x)^2+Q(x)^2$. This gives $P(0)^2+Q(0)^2=f(0)=1000$. Since a perfect square leaves one of $0,1$ as a remainder upon division by 4, it follows that $P(0), Q(0)$ are even. Consequently, the coefficient of x in the polynomial $f(x)=P(x)^2+Q(x)^2$ is a multiple of 4.

Since $f(x)+2x$ is a Fermat polynomial, there exist polynomialss $R(x)$ and $S(x)$ with integer coefficients such that $f(x)+2x=R(x)^2+S(x)^2$. Using a similar argugemt as above, it follows that the coefficient of x in the polynomial $f(x)+2x=R(x)^2+S(x)^2$ is a multiple of 4.

Note that

$$2x=P(x)^2+Q(x)^2-R(x)^2-S(x)^2,$$

and the coefficient of x in $P(x)^2+Q(x)^2-R(x)^2-S(x)^2$ is a multiple of 4. This contradicts the assumption that $f(x)+2x$ is a Fermat polynomial.

This proves that $f(x)+2x$ is not a Fermat polynomial. ■

Exercise 4 (RMO 2013c P6,). Let $n\geq 4$ be a natural number. Let $A_1A_2\cdots A_n$ be a regular polygon and $X=\{1,2,3,\ldots,n\}$. A subset $\{i_1,i_2,\ldots,i_k\}$ of X, with $k\geq 3$ and $i_1<i_2<\cdots<i_k$, is called a *good subset* if the angles of the polygon $A_{i_1}A_{i_2}\cdots A_{i_k}$, when arranged in the increasing order, are in an arithmetic progression. If n is a prime, show that a proper good subset of X contains exactly four elements.

Walkthrough —

(a) Note that the angle subtended by any side of a regular n-gon at the center of its circumcircle is equal to $2\pi/n$.

(b) Show that the angle, subtended by any arc corresponding to two consecutive vertices of a polygon of the form $A_{i_1}A_{i_2}\ldots A_{i_k}$, is a multiple of $2\pi/n$.

(c) Conclude that the angles of a polygon of the form $A_{i_1}A_{i_2}\ldots A_{i_k}$ is a multiple of π/n.

Solution 4. Note that the angle subtended by any side of the regular n-gon $A_1A_2\cdots A_n$ at the center of its circumcircle is equal to $2\pi/n$. Let $k\geq 3$ be a positive integer, and $\{i_1,i_2,\ldots,i_k\}$ be a proper good subset of $\{1,2,\ldots,n\}$, with $i_1<i_2<\cdots<i_k$.

The content posted here and at this blog by Evan Chen are quite useful.

This shows that the angle, subtended by any arc corresponding to two consecutive vertices of the polygon $A_{i_1}A_{i_2}\dots A_{i_k}$, is a multiple of $2\pi/n$. Since the angle subtended by an arc of a circle at its center, is twice of the angle subtended by it at any point on the circumference, it follows that the angles of the polygon are multiples of π/n. Since $\{i_1, i_2, \dots, i_k\}$ is a good subset, and the angles of a k-gon add up to $(k-2)\pi$, it follows that there are k integers $n_1 \le n_2 \le \cdots \le n_k$, which are in arithmetic progression, and their sum is equal to $(k-2)n$. This shows that $k(n_1+n_k)/2 = (k-2)n$, which implies that k divides $2(k-2)n$. Since $\{i_1, i_2, \dots, i_k\}$ is a proper subset of $\{1, 2, \dots, n\}$, and n is a prime, it follows that the integers k, n are relatively prime. This gives that k divides $2(k-2)$, and consequently, k divides 4. Using the bound $k \ge 3$, we obtain $k = 4$. ■