RMO 2013b

CRMO 2

Exercise 1 (RMO 2013b P1,). Prove that there do not exist natural numbers x and y with $x > 1$ such that

$$\frac{x^7 - 1}{x - 1} = y^5 + 1.$$

Solution 1. Let x, y be natural numbers with $x \neq 1$ and satisfying the above equation. Note that the given equation can be rewritten as

$$y^5 = \frac{x^7 - x}{x - 1} = x(x^2 + x + 1)(x^3 + 1).$$

Observe that the integer x is coprime to $x^2 + x + 1$, and x is also coprime to $x^3 + 1$. Since

$$x^3 + 1 - (x - 1)(x^2 + x + 1) = 2$$

holds and the integer x^2+x+1 is odd, it follows that the integers x^2+x+1, x^3+1 are coprime. Since the positive integers $x, x^2 + x + 1, x^3 + 1$ are pairwise coprime, and their product is the fifth power of an integer, it follows that the integers $x, x^2 + x + 1, x^3 + 1$ are also fifth powers of positive integers. Using $1 = (x^3 + 1) - (x^3)$, we obtain that 1 can be expressed as the difference of the fifth powers of two distinct positive integers. However, this is impossible since the fifth powers of two distinct positive integers differ by at least 31. Indeed, if $i > j \geq 1$ are positive integers, then

$$\begin{aligned} i^5 - j^5 &= (i - j)(i^4 j + i^3 j + i^2 j^2 + ij^3 + j^4) \\ &\geq i^4 j + i^3 j + i^2 j^2 + ij^3 + j^4 \\ &\geq 2^4 + 2^3 + 2^2 + 2 + 1 \\ &= 31 \end{aligned}$$

holds. This completes the proof. ■

Remark. It is worth comparing the above problem, and IMOSL 2006 N5 **??**.

Exercise 2 (RMO 2013b P3,). Consider the expression

$$2013^2 + 2014^2 + 2015^2 + \cdots + n^2.$$

Prove that there exists a natural number $n > 2013$ for which one can change a suitable number of plus signs to minus signs in the above expression to make the resulting expression equal 9999.

Summary — "Differentiating" a polynomial enough times makes it linear.

Walkthrough —

(a) Consider the polynomial $P(k) = k^2$, and the polynomial $Q(k) := P(k) - (k-1)$.

(b) Since $Q(k)$ is a linear polynomial in k, the difference $R(k) := Q(k) - Q(k-2)$ is a constant, that is, it does not depend on k.

(c) Note that $R(k)$ is a ± 1-**linear combination**[a] of four consecutive squares.

(d) Does this help?

[a] What does it mean?

Solution 2. Consider the polynomial $P(k) = k^2$, and the polynomial $Q(k) := P(k) - (k-1)$. Since $Q(k)$ is a linear polynomial in k, the difference $R(k) := Q(k) - Q(k-2)$ is a constant, that is, it does not depend on k. Indeed, $Q(k) = 2k-1$, and $R(k) = 4$. Note that $R(k) = k^2-(k-1)^2-(k-2)^2+(k-3)^2$.

Note that

$$2013^2 + 2014^2 + 2015^2 + 2016^2 + 2017^2 > 9999$$

holds, and the integers $2013^2+2014^2+2015^2+2016^2+2017^2, 9999$ are congruent modulo 4, that is, they differ by a multiple of 4. Let $m \geq 1$ be an integer such that

$$9999 = 2013^2 + 2014^2 + 2015^2 + 2016^2 + 2017^2 - 4m$$

holds. Since

$$-k^2 + (k+1)^2 + (k+2)^2 - (k+3)^2 = -4,$$

it follows that

$$\begin{aligned} &9999 \\ &= 2013^2 + 2014^2 + 2015^2 + 2016^2 + 2017^2 \\ &- 2018^2 + 2019^2 + 2020^2 - 2021^2 \end{aligned}$$

The content posted here and at this blog by Evan Chen are quite useful.

$$
\begin{aligned}
&- \dots \\
&- \big((2018 + 4(m-1))^2 + (2019 + 4(m-1))^2 \\
&\qquad + (2020 + 4(m-1))^2 - (2021 + 4(m-1))^2\big).
\end{aligned}
$$

It follows that there exists a natural number $n = 2021 + 4(m-1) > 2013$, for which one can change a suitable number of plus signs to minus signs in the expression

$$2013^2 + 2014^2 + 2015^2 + \dots + n^2$$

to make the resulting expression equal to 9999. ■

Exercise 3 (RMO 2013b P5,). Let $n \geq 3$ be a natural number and let P be a polygon with n sides. Let $a_1, a_2, \dots, a_n$ be the lengths of sides of P and let p be its perimeter. Prove that

$$\frac{a_1}{p - a_1} + \frac{a_2}{p - a_2} + \dots + \frac{a_n}{p - a_n} < 2.$$

Solution 3. By a repeated application of the triangle inequality, it follows that $a_i < p - a_i$ for all $1 \leq i \leq n$. Note that for any positive real numbers a, b, c, the inequality

$$\frac{a}{b} < \frac{a + c}{b + c}$$

holds. It follows that

$$\frac{a_1}{p - a_1} + \frac{a_2}{p - a_2} + \dots + \frac{a_n}{p - a_n} < \frac{2a_1}{p} + \frac{2a_2}{p} + \dots + \frac{2a_n}{p} = 2$$

holds. ■

Exercise 4 (RMO 2013b P6,). For a natural number n, let $T(n)$ denote the number of ways we can place n objects of weights $1, 2, \dots, n$ on a balance such that the sum of the weights in each pan is the same. Prove that $T(100) > T(99)$.

Walkthrough —

(a) Try to find out a placing of the weights $1, 2, \dots, 99$ on a balance such that the weights on the pans are the same.

(b) How about putting $1, 2, 3, \dots, 49$ on one pan, and $51, 52, \dots, 99$ on the other? Note that the weights on the pans are the same. However, the object of weight 50 has not been included. This can be resolved by placing the weights $(\{1, 2, \dots, 49\} \setminus \{25\}) \cup \{50\}$ on one pan, the weights $\{25\} \cup \{51, 52, \dots, 99\}$ on the other.

(c) Does the above help in finding an injective map from the set of all possible placements of the weights $1, 2, \dots, 99$ on a balance satisfying the required condition, to the set of all possible placements of the weights $1, 2, \dots, 100$ on a balance satisfying the required condition? Can one also have in

addition that this map is not surjective?

Solution 4. Let S_n denote the set of pairs (A, B) where A, B are disjoint subsets of $\{1, 2, \ldots, n\}$ such that their union is $\{1, 2, \ldots, n\}$ and the sum of elements of A is equal to the sum of elements of B. To establish $T(99) < T(100)$, it suffices to construct an injective map $f : S_{99} \to S_{100}$ which is not surjective. Define f by

$$f(A, B) = \begin{cases} (A \cup \{100\} \setminus \{50\}, B \cup \{50\}) & \text{if } 50 \in A, \\ (A \cup \{50\}, B \cup \{100\} \setminus \{50\}) & \text{if } 50 \in B. \end{cases}$$

Note that f is well-defined (since given an element (A, B) of S_{99}, either A contains 50 or B contains 50).

Claim — The map f is injective.

Proof of the Claim. Let $(A, B), (C, D)$ be two elements of S_{99} having the same image under f.

Let us consider the case that A contains 50. Since the second coordinate of $f(C, D) = f(A, B)$ contains 50, it follows that C contains 50. Considering the second coordinate of $f(A, B)$ and that of $f(C, D)$, we obtain $B \cup \{50\} = D \cup \{50\}$, which yields $B = D$ (since $50 \notin B$ and $50 \notin D$). This also gives $A = C$.

If B contains 50, then a similar argument shows that $A = C$, and this gives that $B = D$. □

Claim — The map $f : S_{99} \to S_{100}$ is not surjective.

Proof of the Claim. To show that f is not surjective, it suffices to find a pair (A, B) of subsets of $\{1, 2, \ldots, 100\}$ such that $A \cap B = \emptyset$, $A \cup B = \{1, 2, \ldots, 100\}$, and A contains 50 and 100.

Note that for the sum of the elements of any of the following 50 sets

$$\{1, 100\}, \{2, 99\}, \{3, 98\}, \ldots, \{50, 51\}$$

is equal to 101. So the pair (A, B) belongs to S_{100}, where

$$A = \{1, 100\} \cup \{2, 99\} \cup \cdots \cup \{24, 77\} \cup \{50, 51\},$$
$$B = \{26, 75\} \cup \{27, 74\} \cup \cdots \cup \{49, 52\} \cup \{25, 76\}.$$

Moreover, the pair (A, B) does not belong to $f(S_{99})$. □

Using the above Claims, it follows that $T(99) < T(100)$. ■