RMO 2013a

CRMO 1

Exercise 1 (RMO 2013a P2,). Let $f(x) = x^3 + ax^2 + bx + c$ and $g(x) = x^3 + bx^2 + cx + a$, where a, b, c are integers with $c \neq 0$. Suppose that the following conditions hold:

(a) $f(1) = 0$,

(b) the roots of $g(x) = 0$ are the squares of the roots of $f(x) = 0$.

Find the value of $a^{2013} + b^{2013} + c^{2013}$.

Solution 1. Condition (a) implies that 1 is a root of $f(x)$. Let α, β denote the remaining roots of $f(x)$ in $\mathbb{C}$. It follows that

$$\alpha + \beta + 1 = -a, \alpha + \beta + 1 = b, \alpha\beta = -c.$$

Condition (b) shows that the roots of the polynomial $g(x)$ are $1, \alpha^2, \beta^2$. Consequently,

$$\alpha^2 + \beta^2 + 1 = -b, \alpha^2 + \beta^2 + \alpha^2\beta^2 = c, \alpha^2\beta^2 = -a.$$

hold. Note that

$$\begin{aligned} a &= -(\alpha + \beta + 1) \\ &= -b, \\ -b &= (\alpha + \beta + 1)^2 - 2(\alpha + \beta + 1) \\ &= a^2 - 2b \\ &= b^2 - 2b, \end{aligned}$$

which shows that $b^2 = b$, implying $b = 0$ or $b = 1$. Also note that $-a = c^2$. If $b = 0$, then $a = 0$ and hence, $c = 0$. By the given condition, we have $c \neq 0$. It follows that $b = 1$ and hence $a = -1$. Using $f(1) = 0$ combined with $a = -1, b = 1$, we obtain $c = -1$. This gives

$$a^{2013} + b^{2013} + c^{2013} = -1.$$

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Exercise 2 (RMO 2013a P3,). Find all primes p and q such that p divides q^2-4 and q divides p^2-1.

Solution 2. Let p, q be primes such that p divides q^2-4 and q divides p^2-1. If p is equal to 2, then using that p divides q^2-4, it follows that $q=2$, which is impossible since q divides p^2-1. This shows that p is odd.

Since p divides q^2-4, it follows that p divides $q-2$, or p divides $q+2$. Using the hypothesis that q divides p^2-1, we obtain q divides at least one of $p-1, p+1$, and hence, $q \leq p+1$ holds. If p divides $q-2$, then using $q-2 \leq p-1$, we obtain that $q-2=0$, that is, $q=2$. If p divides $q+2$, then using $0 < q+2 \leq p+3$, we obtain $q+2=p$, and using that q divides $p^2-1=(q+2)^2-1=q^2+4q+3$, we get $q=3$, and hence, $p=5$. This proves that (p,q) is equal to $(5,3)$, or that p is odd, and $q=2$.

Note that if (p,q) is equal to $(5,3)$, then the required divisibility conditions hold. Moreover, these conditions are also satisfied if p is an odd prime and $q=2$.

It follows that the required pairs of primes (p,q) are precisely the elements of

$$\{(5,3)\} \cup \{(r,2) \mid r \text{ is an odd prime}\}.$$

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Exercise 3 (RMO 2013a P4,). Find the number of 10-tuples $(a_1, a_2, \ldots, a_9, a_{10})$ of integers such that $|a_1| \leq 1$ and

$$a_1^2+a_2^2+a_3^2+\cdots+a_{10}^2-a_1a_2-a_2a_3-a_3a_4-\cdots-a_9a_{10}-a_{10}a_1=2.$$

Solution 3. Let $(a_1, \ldots, a_{10})$ be a tuple of integers satisfying $|a_1| \leq 1$, and the above equation. Note that the above equation can be rewritten as

$$(a_1-a_2)^2+(a_2-a_3)^2+\cdots+(a_9-a_{10})^2+(a_{10}-a_1)^2=4.$$

Since the sum of the integers

$$a_1-a_2, a_2-a_3, \ldots, a_{10}-a_1$$

is zero, it is not possible that only one of them is equal to ± 2 and the others are zero. Consequently, exactly two of them are equal to 1, and exactly two of them are equal to -1, and the remaining ones are equal to zero. Note that a_1 is equal to $-1, 0$ or 1.

Let $\mathcal{A}$ denote the set of solutions satisfying the given conditions, and $\mathcal{B}$ denote the set of 11-tuples of integers, whose first coordinate is at most 1 in absolute value, the remaining 10 coordinates add up to 0, and exactly two of these 10 coordinates are equal to 1, and exactly two of these 10 coordinates are equal to -1. Consider the map from $\mathcal{A} \to \mathcal{B}$, given by

$$(a_1, \ldots, a_{10}) \mapsto (a_1, a_1-a_2, a_2-a_3, \ldots, a_{10}-a_1).$$

The content posted here and at this blog by Evan Chen are quite useful.

Note that this map is a bijection. So the number of 10-tuples satisfying the given conditions is equal to

$$3\binom{10}{2}\binom{8}{2} = 3780.$$

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Exercise 4 (RMO 2013a P6,). Suppose that m and n are integers such that both the quadratic equations $x^2 + mx - n = 0$ and $x^2 - mx + n = 0$ have integer roots. Prove that n is divisible by 6.

Solution 4. Since the given quadratic polynomials have integer roots, their discriminants are a perfect squares. It follows that there are nonnegative integers a, b satisfying $m^2 + 4n = a^2, m^2 - 4n = b^2$. This gives

$$2m^2 = a^2 + b^2, 8n = a^2 - b^2.$$

Note that if 3 divides m, then 3 divides a and b, and hence, 3 divides n. Also note that if $m \equiv \pm 1 \pmod 3$ holds, then a^2, b^2 are congruent to 1 modulo 3, and hence, 3 divides n.

If m is an odd integer, then a is odd, and hence, m^2, a^2 are congruent to 1 modulo 8, which gives that 8 divides $4n$, implying n is even. If m is even, then so are the integers a and b, and we have

$$2n = \left(\frac{a}{2}\right)^2 - \left(\frac{b}{2}\right)^2,$$

which implies that the integers $a/2, b/2$ are of the same parity, and hence, 4 divides $(a/2)^2 - (b/2)^2$, implying that n is even.

We conclude that n is divisible by 6. ■