RMO 2012f

Mumbai

Exercise 1 (RMO 2012f P1,). Find nonzero real numbers a, b such that $x^2 + ax + b, x^2 + x + ab, ax^2 + x + b$ are three distinct polynomials with a common root.

Solution 1. Let a, b be real numbers such that $x^2+ax+b, x^2+x+ab, ax^2+x+b$ are three distinct polynomials with a common root $\alpha \in \mathbb{C}$. We obtain

$$\alpha^2 + a\alpha + b = \alpha^2 + \alpha + ab = a\alpha^2 + \alpha + b = 0,$$

which gives $a\alpha + b = \alpha + ab$, that is, $(a-1)(\alpha - b) = 0$. Since $x^2 + ax + b, x^2 + x + ab, ax^2 + x + b$ are distinct, it follows that $a \neq 1$. This shows that $\alpha = b$. Since the polynomials $x^2 + x + ab, ax^2 + x + b$ vanish at $x = \alpha = b$, we obtain

$$b(a + b + 1) = b(ab + 2) = 0.$$

Using b is nonzero, we get $a + b + 1 = ab + 2 = 0$. Note that $1 + ab - a - b = 0$. Since $a \neq 1$, we obtain $b = 1$, which combined with $ab + 2 = 0$ implies that $a = -2$.

Also note that for $a = -2, b = 1$, the given polynomials are equal to

$$x^2 - 2x + 1, x^2 + x - 2, -2x^2 + x + 1,$$

which are all distinct and they vanish at $x = 1$.

We conclude that precisely for $(a, b) = (-2, 1)$, the given polynomials are all distinct and have a common root. ■

Exercise 2 (RMO 2012f P2,). Let n be a positive integer such that 13 divides $n^2 + 3n + 51$. Show that 169 divides $21n^2 + 89n + 44$.

Solution 2. Note that the polynomial $X^2+3X+51$ is congruent to X^2+3X-1 modulo 13, which vanishes at 5 mod 13, and its discriminant is congruent to 0 modulo 13. This shows that the polynomial $X^2 + 3X + 51$ is congruent to $(X - 5)^2$ modulo 13. So n is congruent to 5 mod 13.

Note that the discriminant of $21X^2 + 89X + 44$ satisfies

$$89^2 - 4 \cdot 21 \cdot 44 \equiv (-2)^2 - 4 \cdot (-5) \cdot 5 \equiv 4 \cdot 26 \bmod 13,$$

and hence this polynomial is congruent to $21(X-k)^2$ modulo 13 for some integer k. Observing that

$$21 \cdot 5^2 + 89 \cdot 5 + 44 \equiv (-5) \cdot (-1) - 2 \cdot 5 + 5 \equiv 0 \bmod 13,$$

it follows that k is congruent to 5 mod 13. This implies that $21X^2 + 89X + 44$ is congruent to $21(X-5)^2$ modulo 13. Since n is congruent to 5 mod 13, it follows that 169 divides $21n^2 + 89n + 44$. ■

Exercise 3 (RMO 2012f P3,). Solve

$$2^{5x+2[x]} = 11 \cdot 2^{5x} + 5 \cdot 2^{7[x]}$$

for $x \in \mathbb{R}, \mathbb{Q}$.

Walkthrough — Show that

$$2^{5\{x\}} = \frac{5 \cdot 2^{2[x]}}{2^{2[x]} - 11}$$

holds, where $\{x\}$ denotes the fractional part of x. For any real x satisfying the above, prove that $[x] \geq 2$ holds and the RHS of the above lies between 1 and 2^5.

Solution 3. Let x be a real number satisfying the above equation. Denote by $\{x\}$ the fractional part of x, that is $\{x\}$ is equal to $x - [x]$. Replacing x by $[x] + \{x\}$ in the given equation yields

$$2^{7[x]+5\{x\}} = 11 \cdot 2^{5[x]+5\{x\}} + 5 \cdot 2^{7[x]},$$

which implies that $2^{2[x]+5\{x\}} = 11 \cdot 2^{5\{x\}} + 5 \cdot 2^{2[x]}$, which gives

$$2^{5\{x\}} = \frac{5 \cdot 2^{2[x]}}{2^{2[x]} - 11}.$$

Since $2^{5\{x\}}$ is positive, it follows that $2^{2[x]} - 11$ is positive, and hence $[x] \geq 2$. Note that for any integer $n \geq 2$,

$$5 < \frac{5 \cdot 2^{2n}}{2^{2n} - 11} = 5\left(1 + \frac{11}{2^{2n} - 11}\right) \leq 5\left(1 + \frac{11}{5}\right) = 16 < 2^5,$$

holds, and hence $\frac{5 \cdot 2^{2^n}}{2^{2n} - 11}$ is equal to 2^{5y} for some $0 < y < 1$. It follows that

$$x = [x] + \log_{32} \frac{5 \cdot 2^{2[x]}}{2^{2[x]} - 11}, \text{ with } 0 < \log_{32} \frac{5 \cdot 2^{2[x]}}{2^{2[x]} - 11} < 1.$$

It also follows that for any integer $n \geq 2$,

$$0 < \log_{32} \frac{5 \cdot 2^{2n}}{2^{2n} - 11} < 1$$

The content posted here and at this blog by Evan Chen are quite useful.

holds, and hence $n + \log_{32} \frac{5 \cdot 2^{2n}}{2^{2n}-11}$ satisfies the given equation. Consequently, the real solutions are of the form $n + \log_{32} \frac{5 \cdot 2^{2n}}{2^{2n}-11}$ for some integer $n \geq 2$.

Note that if $n + \log_{32} \frac{5 \cdot 2^{2n}}{2^{2n}-11}$ is rational for some integer $n \geq 2$, then $r := \log_{32} \frac{5 \cdot 2^{2n}}{2^{2n}-11}$ is rational and $2^{5r} = \frac{5 \cdot 2^{2n}}{2^{2n}-11}$ is rational. By the fundamental theorem of arithmetic, it follows that $5r = 2n$. Since r lies between 0 and 1, we obtain $r = 4/5$. Substituting $r = 4/5$ in $2^{5r} = \frac{5 \cdot 2^{2n}}{2^{2n}-11}$, we get $n = 2$. So, any rational solution to the given equation is equal to $2 + 4/5$. Also note that $2 + 4/5$ satisfies the given equation. Thus the only rational solution is $2 + \frac{4}{5} = \frac{14}{5}$. ■

Exercise 4 (RMO 2012f P5,). Determine all positive integers a, b, c such that $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$, $a \leq b \leq c$ and a is a prime.

Solution 4. Let a, b, c be integers satisfying the given conditions. Since

$$1 = \frac{1}{a} + \frac{2}{b} + \frac{3}{c} \leq \frac{6}{a}$$

holds, we obtain $a \leq 6$. Since a is a prime, it is equal to one of $2, 3, 5$. Note that

$$\frac{2}{b} < \frac{2}{b} + \frac{3}{c} \leq \frac{5}{b}$$

holds, which implies that

$$\frac{2}{b} < 1 - \frac{1}{a} \leq \frac{5}{b},$$

which yields that

$$\frac{2}{1 - \frac{1}{a}} < b \leq \frac{5}{1 - \frac{1}{a}}.$$

It follows that $5 \leq b \leq 10$ if $a = 2$, $4 \leq b \leq 7$ if $a = 3$, and $b \leq 6$ if $a = 5$. Note that if (a, b) is equal to

$$(2, 9), (3, 5), (3, 7), (5, 5), (5, 6),$$

then no integer c satisfies the given equation. This shows that (a, b, c) is equal to one of

$$(2, 5, 30), (2, 6, 18), (2, 7, 14), (2, 8, 12), (2, 10, 10), (3, 4, 18), (3, 6, 9).$$

Noting that the above triples satisfy the given equation, we conclude that the required solutions are precisely the ones above. ■

Exercise 5 (RMO 2012f P6,). Let S be the set $\{1, 2, \ldots, 10\}$. Let A be a subset of S. We arrange the elements of A in increasing order, that is, $A = \{a_1, a_2, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$. Define WSUM for this subset as $3(a_1 + a_3 + \ldots) + 2(a_2 + a_4 + \ldots)$ where the first term contains the odd numbered terms and the second the even numbered terms. (For example, if $A = \{2, 5, 7, 8\}$, WSUM is $3(2 + 7) + 2(5 + 8)$.) Find the sum of WSUMs over all the subsets of S. (Assume that WSUM for the null set is 0.)

Walkthrough —

(a) Note that

$$\text{WSUM}(\{2,5,7,8\}) = 3(2+7)+2(5+8),$$
$$\text{WSUM}(\{1,2,5,7,8\}) = 3(1+5+8)+2(2+7),$$

which shows that

$$\text{WSUM}(\{2,5,7,8\}) + \text{WSUM}(\{1,2,5,7,8\}) = 3+5(2+5+7+8).$$

(b) Also note that the sum of all the elements of a subset of A of $\{2,3,\ldots,10\}$, and the sum of all the elements of its complement in $\{2,3,\ldots,10\}$, add up to the sum of all the elements of $\{2,3,\ldots,10\}$.

Solution 5. Note that the subsets of $\{1,2,\ldots,10\}$ can be obtained by considering the subsets of $\{2,3,\ldots,10\}$, along with the union of these subsets with $\{1\}$. Using this observation, we decompose the required sum as follows.

$$\begin{aligned}
\sum_{A\subseteq S} \text{WSUM}(A) &= \sum_{A\subseteq\{2,3,\ldots,10\}} \text{WSUM}(A) + \text{WSUM}(A\cup\{1\}) \\
&= \sum_{A\subseteq\{2,3,\ldots,10\}} \left(3+5\sum_{a\in A} a\right) \\
&= 3\cdot 2^9 + 5 \sum_{A\subseteq\{2,3,\ldots,10\}} \sum_{a\in A} a \\
&= 3\cdot 2^9 + \frac{5}{2} \sum_{A\subseteq\{2,3,\ldots,10\}} \left(\sum_{a\in A} a + \sum_{a\in A^c} a\right) \\
&= 3\cdot 2^9 + \frac{5}{2} \sum_{A\subseteq\{2,3,\ldots,10\}} (2+3+\cdots+10) \\
&= 3\cdot 2^9 + \frac{5}{2}\times 54\times 2^9 \\
&= 138\times 2^9.
\end{aligned}$$

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Exercise 6 (RMO 2012f P8,). Let x,y,z be positive real numbers such that $2(xy+yz+zx)=xyz$. Prove that

$$\frac{1}{(x-2)(y-2)(z-2)} + \frac{8}{(x+2)(y+2)(z+2)} \le \frac{1}{32}.$$

If a,b,c are positive real numbers such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$, then show that

$$\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)} \le \frac{1}{4}.$$

The content posted here and at this blog by Evan Chen are quite useful.

Solution 6. Put

$$x = 2a, y = 2b, z = 2c.$$

Note that a, b, c are positive real numbers and they satisfy $ab + bc + ca = abc$. Observe that

$$\begin{aligned}(a-1)(b-1)(c-1) &= abc - ab - bc - ca + a + b + c - 1\\ &= a + b + c - 1,\\ (a+1)(b+1)(c+1) &= abc + ab + bc + ca + a + b + c + 1\\ &= 2abc + a + b + c + 1.\end{aligned}$$

Using the AM-GM-HM inequality, we obtain

$$\frac{a+b+c}{3} \geq (abc)^{\frac{1}{3}} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}},$$

which yields $a + b + c \geq 9$ and $abc \geq 3^3$. This implies that

$$\begin{aligned}&\frac{1}{(a-1)(b-1)(c-1)} + \frac{8}{(a+1)(b+1)(c+1)}\\ &\leq \frac{1}{9-1} + \frac{8}{2\cdot 27 + 9 + 1}\\ &= \frac{1}{8} + \frac{8}{64}\\ &= \frac{1}{4}.\end{aligned}$$

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