RMO 2012e

AoPS

Exercise 1 (RMO 2012e P2,). cf. [**GA17**, Problem 141] Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 3$. Knowing that $a_{n-1} = -\binom{n}{1}$ and $a_{n-2} = \binom{n}{2}$, and that all the roots of P are real, find the remaining coefficients. Note that $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

Solution 1. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the roots of P. Note that

$$\begin{aligned}
&\sum_{1\leq i,j\leq n, i\neq j} (\alpha_i - \alpha_j)^2 \\
&= 2(n-1)\sum_{1\leq i\leq n} \alpha_i^2 - 2\sum_{1\leq i,j\leq n, i\neq j} \alpha_i\alpha_j \\
&= 2(n-1)\sum_{1\leq i\leq n} \alpha_i^2 - 4\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2(n-1)\left(\sum_{1\leq i\leq n} \alpha_i\right)^2 - 4(n-1)\left(\sum_{1\leq i<j\leq n} \alpha_i\alpha_j\right) - 4\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2n^2(n-1) - 4n\sum_{1\leq i<j\leq n} \alpha_i\alpha_j \\
&= 2n^2(n-1) - 4n\binom{n}{2} \\
&= 0.
\end{aligned}$$

Since $\alpha_1, \ldots, \alpha_n$ are real, it follows that they are all equal. Using $\alpha_1 + \cdots + \alpha_n = n$, we get

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1.$$

This implies that $a_i = (-1)^i\binom{n}{i}$ for any $0 \leq i \leq n-1$. ■

Solution 2. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the roots of $P(x)$. Note that

$$\begin{aligned}
&(\alpha_1 - 1)^2 + (\alpha_2 - 1)^2 + \cdots + (\alpha_n - 1)^2 \\
&= \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 + n - 2(\alpha_1 + \alpha_2 + \cdots + \alpha_n)
\end{aligned}$$

$$= (\alpha_1 + \alpha_2 + \cdots + \alpha_n)^2 - 2 \sum_{1 \le i < j \le n} \alpha_i \alpha_j + n - 2(\alpha_1 + \alpha_2 + \cdots + \alpha_n)$$
$$= n^2 - n(n-1) + n - 2n$$
$$= 0.$$

So the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all equal to 1. This implies that $P(x) = (x-1)^n$, and hence a_i is equal to $(-1)^i \binom{n}{i}$ for any $0 \le i \le n$. ■

Exercise 2 (RMO 2012e P3,). Find all natural numbers x, y, z such that

$$(2^x - 1)(2^y - 1) = 2^{2^z} + 1.$$

Solution 3. Let x, y, z be natural numbers satisfying the above equation. Note that the above equation can be rewritten as

$$2^{x+y} = 2^x + 2^y + 2^{2^z}.$$

Let us first consider the case that $x \le y$. Note that $2^z \ge x$ holds and we have

$$2^y = 1 + 2^{y-x} + 2^{2^z - x}. \tag{1}$$

Let us consider the case that $x = y$. Then we obtain

$$2^y = 2 + 2^{2^z - x},$$

which gives $y = 2, 2^z - x = 1$, which yields

$$2^z = x + 1 = y + 1 = 3,$$

which is impossible. This shows that $x < y$.

Let us consider the case that $x < y$. Using Eq. (1), we obtain $2^z = x$, $y - x = 1$ and $y = 2$, which gives

$$x = 1, y = 2, z = 0.$$

It follows that any solution of the given equation in the natural numbers satisfying $x \le y$ is equal to $(1, 2, 0)$. Since the given equation is symmetric in x, y, it follows that any given solution of this equation is equal to one of

$$(1, 2, 0), (2, 1, 0).$$

Note that the above triples satisfy the given equation. Hence, the above triples are precisely all the solutions of the given equation in the natural numbers. ■

Exercise 3 (RMO 2012e P4,). Let a, b, c be positive real numbers such that $abc(a + b + c) = 3$. Prove that we have

$$(a + b)(b + c)(c + a) \ge 8.$$

Also determine the case of equality.

Solution 4. Note that

$$\begin{aligned}(a+b)(b+c)(c+a) &= (a+b+c)(ab+bc+ca) - abc \\ &= abc(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - abc \\ &= 3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - abc \\ &\geq 9\frac{1}{\sqrt[3]{abc}} - abc.\end{aligned}$$

Also note that

$$\begin{aligned}3 &= abc(a+b+c) \\ &\geq 3(abc)^{4/3},\end{aligned}$$

which gives $abc \leq 1$. Using the above inequalities, we obtain

$$\begin{aligned}(a+b)(b+c)(c+a) &\geq 9-1 \\ &= 8.\end{aligned}$$

This proves that

$$(a+b)(b+c)(c+a) \geq 8,$$

where equality holds if and only if all the prior inequalities are equalities, or equivalently, $a=b=c$. Using $abc(a+b+c)=3$, it follows that a,b,c are equal if and only if they are equal to 1. This shows that $(a+b)(b+c)(c+a)=8$ if and only if a,b,c are equal to 1. ■

Exercise 4 (RMO 2012e P6,). A computer program generated 175 positive integers at random, none of which had a prime divisor greater than 10. Prove that there are three numbers among them whose product is the cube of an integer.

Solution 5. Denote these integers by

$$2^{a_1}3^{b_1}5^{c_1}7^{d_1}, 2^{a_2}3^{b_2}5^{c_2}7^{d_2}, \cdots, 2^{a_{175}}3^{b_{175}}5^{c_{175}}7^{d_{175}}.$$

By pigeonhole principle, there exists a subset A of $\{1,2,\ldots,175\}$ of size 59 such that the integers a_i, for i in A, are congruent to each other modulo 3. By the same principle, it follows that there exists a subset B of A of size 20 such that the integers b_i, for i in B, are congruent to each other modulo 3. Applying the same principle, we obtain a subset of C of B of size 7 such that the integers c_i, for i in C, are congruent to each other modulo 3. Using the principle principle once again, we get a subset D of C of size 3 such that the integers d_i, for i in D, are congruent to each other modulo 3. Since $D \subseteq C \subseteq B \subseteq A$ holds, it follows that the product of the three integers

$$2^{a_i}3^{b_i}5^{c_i}7^{d_i}, i \in D$$

is a perfect cube. ■

References

[**GA17**] Răzvan Gelca and Titu Andreescu. *Putnam and beyond*. Second. Springer, Cham, 2017, pp. xviii+850. ISBN: 978-3-319-58986-2; 978-3-319-58988-6. DOI: 10.1007/978-3-319-58988-6. URL: https://doi.org/10.1007/978-3-319-58988-6 (cited p. 1)