RMO 2012d

CRMO 4

Exercise 1 (RMO 2012d P1,). Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as endpoints of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as endpoints of the arc. Inscribe a circle Γ touching the arc AC externally, the arc BD internally and also touching the side AD. Find the radius of the circle Γ.

Solution 1. Let O denote the centre of the circle Γ, and F denote the point of tangency of Γ with the side AD. Note that OF is perpendicular to AD, and hence OF is parallel to AB. Let E denote the foot of the perpendicular from O to BC. Note that OE is parallel to AB. It follows that E, O, F are collinear, and hence

$$EF = EO + OF.$$

Since EF, AB are parallel and so are E, AF, it follows that $ABEF$ is a parallelogram, and hence, we obtain $EF = AB, BE = AF$.

Let O denote the centre of the circle Γ. Let Q denote the point of tangency of Γ with the arc BD, and R denote the point of tangency of Γ with the arc AC. Since Γ and the arc BD touch at Q, it follows that AQ, QO are perpendicular to their common tangent at Q. Hence, the points A, Q, O are collinear, and this gives

$$AO = AQ - OQ.$$

Similarly, since Γ and the arc AC touch at R, it follows that BR, RO are perpendicular to their common tangent at R. Hence, the points B, R, O are collinear, and this gives

$$BO = BR + RO.$$

Let r denote the radius of Γ, and s denote the length of the sides of $ABCD$. It follows from the above that

$$BO = BR + RO = s + r, AO = AQ - OQ = s - r.$$

Applying the Pythagoras' theorem to $\triangle AOF, \triangle BEO$, we obtain

$$BE^2 + OE^2 = (s + r)^2, AF^2 + OF^2 = (s - r)^2.$$

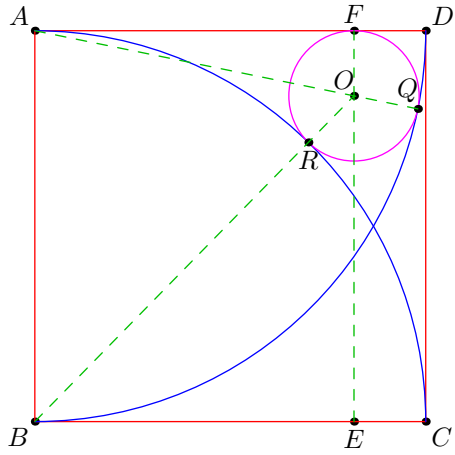


Figure 1: RMO 2012d P1, Exercise 1

Using $BE = AF$, we get

$$(OE - OF)(OE + OF) = 4sr,$$

which yields

$$(EF - 2 \cdot OF)(EF) = 4sr.$$

This gives

$$(s - 2r)s = 4sr,$$

implying that

$$r = \frac{s}{6} = \frac{1}{6}.$$

■

Exercise 2 (RMO 2012d P2, , cf. RMO 2002 P3, RMO 2012a P2, RMO 2012b P2, RMO 2012c P2)**.** Let a, b, c be positive integers such that a divides b^2, b divides c^2 and c divides a^2. Prove that abc divides $(a + b + c)^7$.

Solution 2. In the following, k denotes one of the integers $2, 3, 4, 5$. Assume that $a \mid b^k, b \mid c^k, c \mid a^k$. Note that if one of a, b, c is equal to 1, then all of them are equal to 1, and then there is nothing to prove. Henceforth, let us assume that a, b, c are larger than 1. The given conditions imply that the set of prime divisors of a, b, c are equal.

Let p be a common prime divisor of a, b, c. Let $p^\alpha, p^\beta, p^\gamma$ denote the highest powers of p dividing a, b, c respectively. The given divisibility conditions imply that

$$\alpha \leq k\beta, \beta \leq k\gamma, \gamma \leq k\alpha.$$

The content posted here and at this blog by Evan Chen are quite useful.

This yields

$$\alpha+\beta+\gamma \le \alpha + k^2\alpha + 2\alpha = (k^2+k+1)\alpha,$$

and similarly, $\alpha+\beta+\gamma \le (k^2+k+1)\beta$, $\alpha+\beta+\gamma \le (k^2+k+1)\gamma$, which shows that

$$\alpha+\beta+\gamma \le (k^2+k+1)\min\{\alpha,\beta,\gamma\}.$$

Note that $p^{\min\{\alpha,\beta,\gamma\}}$ divides $a+b+c$. By the above inequality, $p^{\alpha+\beta+\gamma}$ divides $(a+b+c)^{k^2+k+1}$. So abc divides $(a+b+c)^{k^2+k+1}$. Also note that $p^{\alpha+\beta+\gamma}$ is the highest power of p dividing abc. This shows that for each common prime divisor p of a,b,c, the highest power of p that divides abc also divides $(a+b+c)^{k^2+k+1}$. It follows that abc divides $(a+b+c)^{k^2+k+1}$. ■

Solution 3. It suffices to show that abc divides $a^\alpha b^\beta c^\gamma$ for any nonnegative integers α,β,γ such that $\alpha+\beta+\gamma = k^2+k+1$. Note that

$$abc \mid ac^{k+1} \mid a^{k^2+k+1}, \quad abc \mid a^{k+1}b \mid b^{k^2+k+1}, \quad abc \mid b^{k+1}c \mid c^{k^2+k+1},$$

which shows that abc divides $a^{k^2+k+1}, b^{k^2+k+1}, c^{k^2+k+1}$. Moreover, abc divides $a^\alpha b^\beta c^\gamma$ when α,β,γ are all positive. It remains to consider the case when exactly one of α,β,γ is zero, which we assume from now on. If $\alpha = 0$, then

$$abc \mid b^{k+1}c \mid b^\beta c^\gamma \text{ if } \beta \ge k+1,$$
$$abc \mid c^{k^2+1}b \mid b^\beta c^\gamma \text{ if } \beta \le k$$

holds. If $\beta = 0$, then

$$abc \mid c^{k+1}a \mid c^\gamma a^\alpha \text{ if } \gamma \ge k+1,$$
$$abc \mid a^{k^2+1}c \mid c^\gamma a^\alpha \text{ if } \gamma \le k$$

holds. If $\gamma = 0$, then

$$abc \mid a^{k+1}b \mid a^\alpha b^\beta \text{ if } \alpha \ge k+1,$$
$$abc \mid b^{k^2+1}a \mid a^\alpha b^\beta \text{ if } \alpha \le k$$

holds. This shows that abc divides $(a+b+c)^{k^2+k+1}$. ■

Exercise 3 (RMO 2012d P3, , RMO 2012a P3, RMO 2012b P3, RMO 2012c P3)**.** Let a and b be positive real numbers such that $a+b=1$. Prove that $a^a b^b + a^b b^a \le 1$.

Solution 4. For any positive a,b, the sequences $(a^a, b^a), (a^b, b^b)$ are sorted the same way. Applying the rearrangement inequality, we obtian

$$a^a b^b + a^b b^a \le a^a a^b + b^a b^b = a^{a+b} + b^{a+b} = a+b = 1.$$

■

Exercise 4 (RMO 2012d P4, , cf. RMO 2012a P4, RMO 2012b P4, RMO 2012c P4)**.** Let $X = \{1, 2, 3, \ldots, 11\}$. Find the number of pairs of $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{4, 5, 7, 8, 9, 10\}$.

> **Walkthrough** — Establish a suitable one-to-one correspondence between the set of ordered pairs (A, B) of subsets of $\{1, 2, \ldots, 10\}$ satisfying $A \cap B = \{2, 3, 5, 7\}$, and the set of maps $f : \{1, 2, \ldots, 10\} \setminus \{2, 3, 5, 7\} \to \{0, 1, 2\}$.

Solution 5. Note that given a subset Y of a set X, the ordered pairs (A, B) of subsets of X with $A \cap B = Y$ are in one-to-one correspondence with the maps from $X \setminus Y$ to $\{0, 1, 2\}$ (where the inverse images of $0, 1, 2$ correspond to $A \setminus (A \cap B), B \setminus (A \cap B), X \setminus (A \cup B)$, respectively). Moreover, if (A, B) is such an ordered pair, then $(A, B) = (B, A)$ holds if and only if $A = B = Y$. It follows that the number of ordered pairs $\{A, B\}$ satisfying $A \cap B = Y$ and $A \neq B$ is equal to $\frac{1}{2}(3^{|X \setminus Y|} - 1)$ if $X \setminus Y$ is finite.

The number of pairs of the given type are $3^6 - 1, 3^7 - 1, 3^7 - 1, 3^5 - 1$ for the above four problems, respectively. ■

Exercise 5 (RMO 2012d P6,)**.** Solve the system of equations for positive real numbers:

$$\frac{1}{xy} = \frac{x}{z} + 1,$$
$$\frac{1}{yz} = \frac{y}{x} + 1,$$
$$\frac{1}{zx} = \frac{z}{y} + 1.$$

Solution 6. Let x, y, z be positive real numbers satisfying the given equations. It follows that $z = x^2y + xyz, x = y^2z + xyz, y = z^2x + xyz$, which gives

$$x - y = z(y^2 - zx), y - z = x(z^2 - xy), z - x = y(x^2 - yz).$$

Note that if two of x, y, z are equal, then all of them are equal. Assume that x, y, z are pairwise distinct. Without loss of generality[1], we may assume x is the maximum of x, y, z. If $x > y > z$ holds, then we obtain $y^2 > zx, z^2 > xy, x^2 < yz$, which gives $y^3 > xyz > x^3$ and contradicts $x > y$. On the other hand, $x > z > y$ implies $y^2 > zx, z^2 < xy, x^2 < yz$, which implies $x^3 < xyz < y^3$ and thus contradicts $x > y$. This shows that x, y, z cannot be distinct, and hence they are equal. This yields $x = 2x^3$, which gives $x = y = z = \frac{1}{\sqrt{2}}$. Note that the tuple $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ does satisfy the given system of equations. Hence, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is the only solution of the given system. ■

[1] Is it clear that there is no loss of generality?