# RMO 2012c

## CRMO 3

**Exercise 1** (RMO 2012c P1, ). Let $ABCD$ be a unit square. Draw a quadrant of a circle with $A$ as centre and $B, D$ as endpoints of the arc. Similarly, draw a quadrant of a circle with $B$ as centre and $A, C$ as endpoints of the arc. Inscribe a circle $\Gamma$ touching the arcs $AC$ and $BD$ both externally and also touching the side $CD$. Find the radius of the circle $\Gamma$.

**Solution 1.** Let $E$ denote the midpoint of $AB$, and $F$ denote the midpoint of $CD$. Note that $EF$ is parallel to $AB$ and $CD$, and has length equal to that of $AB$ and $CD$. Indeed, the line passing through the midpoint $E$ of $AB$ and parallel to $BC$ bisects $AC$. Since this line is also parallel to $AD$, it bisects $CD$, hence passes through $F$. Thus, $EF$ is parallel to $AB$ and $CD$, and has length equal to that of $AB$ and $CD$.

Note that $\triangle ADF, \triangle BCF$ are congruent, and hence $AF = BF$. This implies that $\triangle AEF, \triangle BEF$ are congruent, and hence $EF$ is a perpendicular bisector of $AB$. Similarly, it follows that $EF$ is a perpendicular bisector of $CD$.

Let $O$ denote the centre of the circle $\Gamma$. Let $Q$ denote the point of tangency of $\Gamma$ with the arc $BD$, and $R$ denote the point of tangency of $\Gamma$ with the arc $AC$. Since $\Gamma$ and the arc $BD$ touch at $Q$, it follows that $AQ, QO$ are perpendicular to their common tangent at $Q$. Hence, the points $A, Q, O$ are collinear, and this gives

$$AO = AQ + QO.$$

Similarly, since $\Gamma$ and the arc $AC$ touch at $R$, it follows that $BR, RO$ are perpendicular to their common tangent at $R$. Hence, the points $B, R, O$ are collinear, and this gives

$$BO = BR + RO.$$

This implies that

$$BO = BR + RO = BC + QO = AD + QO = AQ + QO = AO.$$

Using an argument as above, it turns out that $\triangle AEO, \triangle BEO$ are congruent, and hence $EO$ is a perpendicular bisector of $AB$. This implies that $O$ lies on the line $EF$.

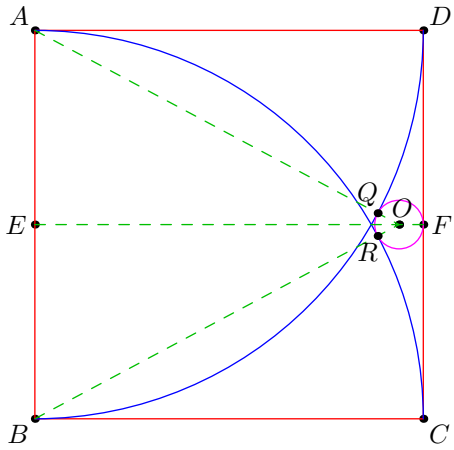



Figure 1: RMO 2012c P1, Exercise 1

Since $\Gamma$ is a circle touching $CD$, it follows that the point of tangency of $\Gamma$ with $CD$ is the foot of the perpendicular from $O$ to $CD$. Since $OF$ is perpendicular to $CD$, it follows that the point of tangency of $\Gamma$ with $CD$ is $F$. Thus, the radius of $\Gamma$ is equal to $OF$. This gives

$$EO = EF - OF.$$

Let $r$ denote the radius of $\Gamma$, and $s$ denote the length of the sides of $ABCD$. It follows from the above that

$$EO = EF - OF = s - r, AO = AQ + QO = s + r.$$

Applying the Pythagoras' theorem to $\triangle AEO$, we obtain

$$(s-r)^2 + \left(\frac{s}{2}\right)^2 = (s+r)^2,$$

which gives $r = \frac{s}{16} = \frac{1}{16}$. ■

**Exercise 2** (RMO 2012c P2, , cf. RMO 2002 P3, RMO 2012d P2, RMO 2012a P2, RMO 2012b P2)**.** Let $a, b, c$ be positive integers such that $a|b^5, b|c^5$ and $c|a^5$. Prove that $abc$ divides $(a+b+c)^{31}$.

**Solution 2.** In the following, $k$ denotes one of the integers $2, 3, 4, 5$. Assume that $a \mid b^k, b \mid c^k, c \mid a^k$. Note that if one of $a, b, c$ is equal to 1, then all of them are equal to 1, and then there is nothing to prove. Henceforth, let us assume that $a, b, c$ are larger than 1. The given conditions imply that the set of prime divisors of $a, b, c$ are equal.



The content posted here and at this blog by Evan Chen are quite useful.

Let $p$ be a common prime divisor of $a, b, c$. Let $p^\alpha, p^\beta, p^\gamma$ denote the highest powers of $p$ dividing $a, b, c$ respectively. The given divisibility conditions imply that

$$\alpha \leq k\beta, \beta \leq k\gamma, \gamma \leq k\alpha.$$

This yields

$$\alpha + \beta + \gamma \leq \alpha + k^2\alpha + 2\alpha = (k^2 + k + 1)\alpha,$$

and similarly, $\alpha + \beta + \gamma \leq (k^2 + k + 1)\beta$, $\alpha + \beta + \gamma \leq (k^2 + k + 1)\gamma$, which shows that

$$\alpha + \beta + \gamma \leq (k^2 + k + 1)\min\{\alpha, \beta, \gamma\}.$$

Note that $p^{\min\{\alpha,\beta,\gamma\}}$ divides $a+b+c$. By the above inequality, $p^{\alpha+\beta+\gamma}$ divides $(a + b + c)^{k^2+k+1}$. So $abc$ divides $(a + b + c)^{k^2+k+1}$. Also note that $p^{\alpha+\beta+\gamma}$ is the highest power of $p$ dividing $abc$. This shows that for each common prime divisor $p$ of $a, b, c$, the highest power of $p$ that divides $abc$ also divides $(a + b + c)^{k^2+k+1}$. It follows that $abc$ divides $(a + b + c)^{k^2+k+1}$. ■

**Solution 3.** It suffices to show that $abc$ divides $a^\alpha b^\beta c^\gamma$ for any nonnegative integers $\alpha, \beta, \gamma$ such that $\alpha + \beta + \gamma = k^2 + k + 1$. Note that

$$abc \mid ac^{k+1} \mid a^{k^2+k+1}, \quad abc \mid a^{k+1}b \mid b^{k^2+k+1}, \quad abc \mid b^{k+1}c \mid c^{k^2+k+1},$$

which shows that $abc$ divides $a^{k^2+k+1}, b^{k^2+k+1}, c^{k^2+k+1}$. Moreover, $abc$ divides $a^\alpha b^\beta c^\gamma$ when $\alpha, \beta, \gamma$ are all positive. It remains to consider the case when exactly one of $\alpha, \beta, \gamma$ is zero, which we assume from now on. If $\alpha = 0$, then

$$abc \mid b^{k+1}c \mid b^\beta c^\gamma \text{ if } \beta \geq k + 1,$$
$$abc \mid c^{k^2+1}b \mid b^\beta c^\gamma \text{ if } \beta \leq k$$

holds. If $\beta = 0$, then

$$abc \mid c^{k+1}a \mid c^\gamma a^\alpha \text{ if } \gamma \geq k + 1,$$
$$abc \mid a^{k^2+1}c \mid c^\gamma a^\alpha \text{ if } \gamma \leq k$$

holds. If $\gamma = 0$, then

$$abc \mid a^{k+1}b \mid a^\alpha b^\beta \text{ if } \alpha \geq k + 1,$$
$$abc \mid b^{k^2+1}a \mid a^\alpha b^\beta \text{ if } \alpha \leq k$$

holds. This shows that $abc$ divides $(a + b + c)^{k^2+k+1}$. ■

**Exercise 3** (RMO 2012c P3, , RMO 2012a P3, RMO 2012b P3, RMO 2012d P3)**.** Let $a$ and $b$ be positive real numbers such that $a + b = 1$. Prove that $a^a b^b + a^b b^a \leq 1$.

**Solution 4.** For any positive $a, b$, the sequences $(a^a, b^a), (a^b, b^b)$ are sorted the same way. Applying the rearrangement inequality, we obtian

$$a^a b^b + a^b b^a \leq a^a a^b + b^a b^b = a^{a+b} + b^{a+b} = a + b = 1.$$

■

**Exercise 4** (RMO 2012c P4, , cf. RMO 2012a P4, RMO 2012b P4, RMO 2012d P4)**.** Let $X = \{1, 2, 3, \ldots, 10\}$. Find the number of pairs of $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{5, 7, 8\}$.

> **Walkthrough** — Establish a suitable one-to-one correspondence between the set of ordered pairs $(A, B)$ of subsets of $\{1, 2, \ldots, 10\}$ satisfying $A \cap B = \{2, 3, 5, 7\}$, and the set of maps $f : \{1, 2, \ldots, 10\} \setminus \{2, 3, 5, 7\} \to \{0, 1, 2\}$.

**Solution 5.** Note that given a subset $Y$ of a set $X$, the ordered pairs $(A, B)$ of subsets of $X$ with $A \cap B = Y$ are in one-to-one correspondence with the maps from $X \setminus Y$ to $\{0, 1, 2\}$ (where the inverse images of $0, 1, 2$ correspond to $A \setminus (A \cap B), B \setminus (A \cap B), X \setminus (A \cup B)$, respectively). Moreover, if $(A, B)$ is such an ordered pair, then $(A, B) = (B, A)$ holds if and only if $A = B = Y$. It follows that the number of ordered pairs $\{A, B\}$ satisfying $A \cap B = Y$ and $A \neq B$ is equal to $\frac{1}{2}(3^{|X \setminus Y|} - 1)$ if $X \setminus Y$ is finite.

The number of pairs of the given type are $3^6 - 1, 3^7 - 1, 3^7 - 1, 3^5 - 1$ for the above four problems, respectively. ■

**Exercise 5** (RMO 2012c P6, )**.** Find all positive integers such that $3^{2n} + 3n^2 + 7$ is a perfect square.

**Solution 6.** Let $n, m$ be positive integers such that

$$3^{2n} + 3n^2 + 7 = m^2$$

holds. This implies

$$3n^2 + 7 = (m - 3^n)(m + 3^n).$$

Note that any two positive integers $a, b$ satisfy $ab > a - b$. Since $m - 3^n, m + 3^n$ are positive, it follows that

$$3n^2 + 7 > 2 \cdot 3^n.$$

Applying the binomial theorem, we obtain

$$3^n \geq 1 + \binom{n}{1} 2 + \binom{n}{2} 2^2 \geq 1 + 2n + 2^2 \frac{n(n-1)}{2} = 1 + 2n^2,$$

which yields

$$2(1 + 2n^2) < 3n^2 + 7,$$

which is equivalent to $n^2 < 5$. This shows that $n$ is equal to 1 or 2. Note that $3^{2n} + 3n^2 + 7$ is equal to 19 (resp. 100) for $n = 1$ (resp. $n = 2$). So the required solution is $n = 2$. ■