RMO 2012b

CRMO 2

Exercise 1 (RMO 2012b P1,). Let $ABCD$ be a unit square. Draw a quadrant of a circle with A as centre and B, D as endpoints of the arc. Similarly, draw a quadrant of a circle with B as centre and A, C as endpoints of the arc. Inscribe a circle Γ touching the arc AC internally, the arc BD internally and also touching the side AB. Find the radius of the circle Γ.

Solution 1. Let E denote the midpoint of AB, and F denote the midpoint of CD. Note that EF is parallel to AB and CD, and has length equal to that of AB and CD. Indeed, the line passing through the midpoint E of AB and parallel to BC bisects AC. Since this line is also parallel to AD, it bisects CD, hence passes through F. Thus, EF is parallel to AB and CD, and has length equal to that of AB and CD.

Note that $\triangle ADF, \triangle BCF$ are congruent, and hence $AF = BF$. This implies that $\triangle AEF, \triangle BEF$ are congruent, and hence EF is a perpendicular bisector of AB.

Let O denote the centre of the circle Γ. Let Q denote the point of tangency of Γ with the arc BD, and R denote the point of tangency of Γ with the arc AC. Since Γ and the arc BD touch at Q, it follows that AQ, QO are perpendicular to their common tangent at Q. Hence, the points A, Q, O are collinear, and this gives

$$AO = AQ - OQ.$$

Similarly, since Γ and the arc AC touch at R, it follows that B, R, O are collinear, and this gives

$$BO = BR - OR.$$

This implies that

$$BR - OR = BC - OR = AD - OR = AQ - OQ = AO.$$

Using an argument as above, it turns out that $\triangle AEO, \triangle BEO$ are congruent, and hence EO is a perpendicular bisector of AB. This implies that O lies on the line EF.

Since Γ is a circle touching AB, it follows that the point of tangency of Γ with AB is the foot of the perpendicular from O to AB. Since OE is perpendicular

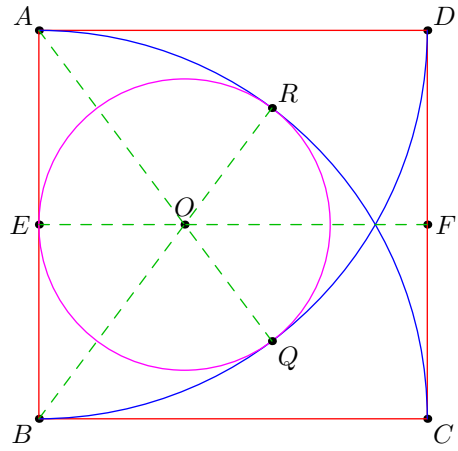


Figure 1: RMO 2012b P1, Exercise 1

to AB, it follows that the point of tangency of Γ with AB is E. Thus, the radius of Γ is equal to OE.

Let r denote the radius of Γ, and s denote the length of the sides of $ABCD$. It follows from the above that

$$EO = r, AO = AQ - OQ = s - r.$$

Applying the Pythagoras' theorem to $\triangle AEO$, we obtain

$$r^2 + \left(\frac{s}{2}\right)^2 = (s-r)^2,$$

which gives $r = \frac{3s}{8} = \frac{3}{8}$. ■

Exercise 2 (RMO 2012b P2, , cf. RMO 2002 P3, RMO 2012d P2, RMO 2012a P2, RMO 2012c P2)**.** Let a, b, c be positive integers such that $a|b^4, b|c^4$ and $c|a^4$. Prove that abc divides $(a+b+c)^{21}$.

Solution 2. In the following, k denotes one of the integers $2, 3, 4, 5$. Assume that $a \mid b^k, b \mid c^k, c \mid a^k$. Note that if one of a, b, c is equal to 1, then all of them are equal to 1, and then there is nothing to prove. Henceforth, let us assume that a, b, c are larger than 1. The given conditions imply that the set of prime divisors of a, b, c are equal.

Let p be a common prime divisor of a, b, c. Let $p^\alpha, p^\beta, p^\gamma$ denote the highest powers of p dividing a, b, c respectively. The given divisibility conditions imply that

$$\alpha \leq k\beta, \beta \leq k\gamma, \gamma \leq k\alpha.$$

This yields

$$\alpha + \beta + \gamma \leq \alpha + k^2\alpha + 2\alpha = (k^2 + k + 1)\alpha,$$

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and similarly, $\alpha+\beta+\gamma \le (k^2+k+1)\beta$, $\alpha+\beta+\gamma \le (k^2+k+1)\gamma$, which shows that

$$\alpha+\beta+\gamma \le (k^2+k+1)\min\{\alpha,\beta,\gamma\}.$$

Note that $p^{\min\{\alpha,\beta,\gamma\}}$ divides $a+b+c$. By the above inequality, $p^{\alpha+\beta+\gamma}$ divides $(a+b+c)^{k^2+k+1}$. So abc divides $(a+b+c)^{k^2+k+1}$. Also note that $p^{\alpha+\beta+\gamma}$ is the highest power of p dividing abc. This shows that for each common prime divisor p of a,b,c, the highest power of p that divides abc also divides $(a+b+c)^{k^2+k+1}$. It follows that abc divides $(a+b+c)^{k^2+k+1}$. ■

Solution 3. It suffices to show that abc divides $a^\alpha b^\beta c^\gamma$ for any nonnegative integers α,β,γ such that $\alpha+\beta+\gamma = k^2+k+1$. Note that

$$abc \mid ac^{k+1} \mid a^{k^2+k+1}, \quad abc \mid a^{k+1}b \mid b^{k^2+k+1}, \quad abc \mid b^{k+1}c \mid c^{k^2+k+1},$$

which shows that abc divides $a^{k^2+k+1}, b^{k^2+k+1}, c^{k^2+k+1}$. Moreover, abc divides $a^\alpha b^\beta c^\gamma$ when α,β,γ are all positive. It remains to consider the case when exactly one of α,β,γ is zero, which we assume from now on. If $\alpha = 0$, then

$$abc \mid b^{k+1}c \mid b^\beta c^\gamma \text{ if } \beta \ge k+1,$$
$$abc \mid c^{k^2+1}b \mid b^\beta c^\gamma \text{ if } \beta \le k$$

holds. If $\beta = 0$, then

$$abc \mid c^{k+1}a \mid c^\gamma a^\alpha \text{ if } \gamma \ge k+1,$$
$$abc \mid a^{k^2+1}c \mid c^\gamma a^\alpha \text{ if } \gamma \le k$$

holds. If $\gamma = 0$, then

$$abc \mid a^{k+1}b \mid a^\alpha b^\beta \text{ if } \alpha \ge k+1,$$
$$abc \mid b^{k^2+1}a \mid a^\alpha b^\beta \text{ if } \alpha \le k$$

holds. This shows that abc divides $(a+b+c)^{k^2+k+1}$. ■

Exercise 3 (RMO 2012b P3, , RMO 2012a P3, RMO 2012c P3, RMO 2012d P3)**.** Let a and b be positive real numbers such that $a+b=1$. Prove that $a^a b^b + a^b b^a \le 1$.

Solution 4. For any positive a,b, the sequences $(a^a, b^a), (a^b, b^b)$ are sorted the same way. Applying the rearrangement inequality, we obtian

$$a^a b^b + a^b b^a \le a^a a^b + b^a b^b = a^{a+b} + b^{a+b} = a+b = 1.$$

■

Exercise 4 (RMO 2012b P4, , cf. RMO 2012a P4, RMO 2012c P4, RMO 2012d P4)**.** Let $X = \{1,2,3,\ldots,12\}$. Find the number of pairs $\{A,B\}$ such that $A \subseteq X, B \subseteq X, A \ne B$ and $A \cap B = \{2,3,5,7,8\}$.

> **Walkthrough** — Establish a suitable one-to-one correspondence between the set of ordered pairs (A, B) of subsets of $\{1, 2, \ldots, 10\}$ satisfying $A \cap B = \{2, 3, 5, 7\}$, and the set of maps $f : \{1, 2, \ldots, 10\} \setminus \{2, 3, 5, 7\} \to \{0, 1, 2\}$.

Solution 5. Note that given a subset Y of a set X, the ordered pairs (A, B) of subsets of X with $A \cap B = Y$ are in one-to-one correspondence with the maps from $X \setminus Y$ to $\{0, 1, 2\}$ (where the inverse images of $0, 1, 2$ correspond to $A \setminus (A \cap B), B \setminus (A \cap B), X \setminus (A \cup B)$, respectively). Moreover, if (A, B) is such an ordered pair, then $(A, B) = (B, A)$ holds if and only if $A = B = Y$. It follows that the number of ordered pairs $\{A, B\}$ satisfying $A \cap B = Y$ and $A \neq B$ is equal to $\frac{1}{2}(3^{|X \setminus Y|} - 1)$ if $X \setminus Y$ is finite.

The number of pairs of the given type are $3^6 - 1, 3^7 - 1, 3^7 - 1, 3^5 - 1$ for the above four problems, respectively. ■

Exercise 5 (RMO 2012b P6,). Show that for all real numbers x, y, z such that $x + y + z = 0$ and $xy + yz + zx = -3$, the expression $x^3y + y^3z + z^3x$ is a constant.

Solution 6. Consider the polynomial

$$P(t) = t^3 - (x + y + z)t^2 + (xy + yz + zx)t - xyz.$$

Since x, y, z are the roots[1] of the equation $P(t) = 0$, we obtain

$$\begin{aligned} x^3 - (x + y + z)x^2 + (xy + yz + zx)x - xyz &= 0, \\ y^3 - (x + y + z)y^2 + (xy + yz + zx)y - xyz &= 0, \\ z^3 - (x + y + z)z^2 + (xy + yz + zx)z - xyz &= 0. \end{aligned}$$

Using them, we obtain

$$\begin{aligned} x^3y + y^3z + z^3x &= \big((x + y + z)x^2 - (xy + yz + zx)x + xyz\big)y \\ &\quad + \big((x + y + z)y^2 - (xy + yz + zx)y + xyz\big)z \\ &\quad + \big((x + y + z)z^2 - (xy + yz + zx)z + xyz\big)x \\ &= (x + y + z)(x^2y + y^2z + z^2x) \\ &\quad - (xy + yz + zx)(xy + yz + zx) \\ &\quad + xyz(x + y + z) \\ &= -(xy + yz + zx)^2 \qquad (\text{using } x + y + z = 0) \\ &= -9 \qquad (\text{using } xy + yz + zx = -3). \end{aligned}$$

This completes the proof. ■

[1] If it is not clear, then the following equalities may directly be verified.

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