RMO 2012a

CRMO 1

Exercise 1 (RMO 2012a P2, , cf. RMO 2002 P3, RMO 2012d P2, RMO 2012b P2, RMO 2012c P2)**.** Let a, b, c be positive integers such that $a|b^3, b|c^3$ and $c|a^3$. Prove that abc divides $(a+b+c)^{13}$.

Solution 1. In the following, k denotes one of the integers $2, 3, 4, 5$. Assume that $a \mid b^k, b \mid c^k, c \mid a^k$. Note that if one of a, b, c is equal to 1, then all of them are equal to 1, and then there is nothing to prove. Henceforth, let us assume that a, b, c are larger than 1. The given conditions imply that the set of prime divisors of a, b, c are equal.

Let p be a common prime divisor of a, b, c. Let $p^\alpha, p^\beta, p^\gamma$ denote the highest powers of p dividing a, b, c respectively. The given divisibility conditions imply that

$$\alpha \leq k\beta, \beta \leq k\gamma, \gamma \leq k\alpha.$$

This yields

$$\alpha + \beta + \gamma \leq \alpha + k^2\alpha + 2\alpha = (k^2 + k + 1)\alpha,$$

and similarly, $\alpha + \beta + \gamma \leq (k^2 + k + 1)\beta$, $\alpha + \beta + \gamma \leq (k^2 + k + 1)\gamma$, which shows that

$$\alpha + \beta + \gamma \leq (k^2 + k + 1)\min\{\alpha, \beta, \gamma\}.$$

Note that $p^{\min\{\alpha,\beta,\gamma\}}$ divides $a+b+c$. By the above inequality, $p^{\alpha+\beta+\gamma}$ divides $(a+b+c)^{k^2+k+1}$. So abc divides $(a+b+c)^{k^2+k+1}$. Also note that $p^{\alpha+\beta+\gamma}$ is the highest power of p dividing abc. This shows that for each common prime divisor p of a, b, c, the highest power of p that divides abc also divides $(a+b+c)^{k^2+k+1}$. It follows that abc divides $(a+b+c)^{k^2+k+1}$. ■

Solution 2. It suffices to show that abc divides $a^\alpha b^\beta c^\gamma$ for any nonnegative integers α, β, γ such that $\alpha + \beta + \gamma = k^2 + k + 1$. Note that

$$abc \mid ac^{k+1} \mid a^{k^2+k+1}, \quad abc \mid a^{k+1}b \mid b^{k^2+k+1}, \quad abc \mid b^{k+1}c \mid c^{k^2+k+1},$$

which shows that abc divides $a^{k^2+k+1}, b^{k^2+k+1}, c^{k^2+k+1}$. Moreover, abc divides $a^\alpha b^\beta c^\gamma$ when α, β, γ are all positive. It remains to consider the case when exactly one of α, β, γ is zero, which we assume from now on. If $\alpha = 0$, then

$$abc \mid b^{k+1}c \mid b^\beta c^\gamma \text{ if } \beta \geq k + 1,$$

$$abc \mid c^{k^2+1}b \mid b^\beta c^\gamma \text{ if } \beta \le k$$

holds. If $\beta = 0$, then

$$abc \mid c^{k+1}a \mid c^\gamma a^\alpha \text{ if } \gamma \ge k+1,$$
$$abc \mid a^{k^2+1}c \mid c^\gamma a^\alpha \text{ if } \gamma \le k$$

holds. If $\gamma = 0$, then

$$abc \mid a^{k+1}b \mid a^\alpha b^\beta \text{ if } \alpha \ge k+1,$$
$$abc \mid b^{k^2+1}a \mid a^\alpha b^\beta \text{ if } \alpha \le k$$

holds. This shows that abc divides $(a+b+c)^{k^2+k+1}$. ■

Exercise 2 (RMO 2012a P3, , RMO 2012b P3, RMO 2012c P3, RMO 2012d P3)**.** Let a and b be positive real numbers such that $a+b=1$. Prove that $a^a b^b + a^b b^a \le 1$.

Solution 3. For any positive a, b, the sequences $(a^a, b^a), (a^b, b^b)$ are sorted the same way. Applying the rearrangement inequality, we obtian

$$a^a b^b + a^b b^a \le a^a a^b + b^a b^b = a^{a+b} + b^{a+b} = a + b = 1.$$

■

Exercise 3 (RMO 2012a P4, , cf. RMO 2012b P4, RMO 2012c P4, RMO 2012d P4)**.** Let $X = \{1, 2, 3, \ldots, 10\}$. Find the number of pairs $\{A, B\}$ such that $A \subseteq X, B \subseteq X, A \ne B$ and $A \cap B = \{2, 3, 5, 7\}$.

> **Walkthrough** — Establish a suitable one-to-one correspondence between the set of ordered pairs (A, B) of subsets of $\{1, 2, \ldots, 10\}$ satisfying $A \cap B = \{2, 3, 5, 7\}$, and the set of maps $f : \{1, 2, \ldots, 10\} \setminus \{2, 3, 5, 7\} \to \{0, 1, 2\}$.

Solution 4. Note that given a subset Y of a set X, the ordered pairs (A, B) of subsets of X with $A \cap B = Y$ are in one-to-one correspondence with the maps from $X \setminus Y$ to $\{0, 1, 2\}$ (where the inverse images of $0, 1, 2$ correspond to $A \setminus (A \cap B), B \setminus (A \cap B), X \setminus (A \cup B)$, respectively). Moreover, if (A, B) is such an ordered pair, then $(A, B) = (B, A)$ holds if and only if $A = B = Y$. It follows that the number of ordered pairs $\{A, B\}$ satisfying $A \cap B = Y$ and $A \ne B$ is equal to $\frac{1}{2}(3^{|X \setminus Y|} - 1)$ if $X \setminus Y$ is finite.

The number of pairs of the given type are $3^6 - 1, 3^7 - 1, 3^7 - 1, 3^5 - 1$ for the above four problems, respectively. ■

The content posted here and at this blog by Evan Chen are quite useful.

Exercise 4 (RMO 2012a P6,). Let a and b be real numbers such that $a \neq 0$. Prove that not all the roots of $ax^4 + bx^3 + x^2 + x + 1 = 0$ can be real.

Solution 5. Since a is nonzero, the polynomial $ax^4 + bx^3 + x^2 + x + 1$ has four roots in $\mathbb{C}$. Denote these roots by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Since $\alpha_1\alpha_2\alpha_3\alpha_4$ is equal to $\frac{1}{a}$, it follows that the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are nonzero. Note that the reciprocals of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of the polynomial $x^4 + x^3 + x^2 + bx + a$. This gives

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} = -1, \sum_{1 \leq i < j \leq 4} \frac{1}{\alpha_i \alpha_j} = 1,$$

which yields

$$\begin{aligned} \frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{1}{\alpha_4^2} &= \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4}\right)^2 - 2 \sum_{1 \leq i < j \leq 4} \frac{1}{\alpha_i \alpha_j} \\ &= -1. \end{aligned}$$

This shows that not all of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are real. ■