RMO 2011b

Exercise 1 (RMO 2011b P2, All-Russian MO 1992–1993 Grade 9 Final Stage P1)**.** Let n be a positive integer such that $2n+1$ and $3n+1$ are both perfect squares. Show that $5n+3$ is a composite number.

Solution 1. Let a, b be positive integers such that

$$2n+1 = a^2, \quad 3n+1 = b^2$$

holds. Note that

$$5n+3 = 4(2n+1) - (3n+1) = 4a^2 - b^2 = (2a-b)(2a+b). \tag{1}$$

Since $2a+b \geq 2$, it follows that $2a-b$ is positive, and hence, it suffices to show that $2a-b \neq 1$. Using $2n+1$ is a perfect square, it follows that $n \geq 3$. Note that

$$\begin{aligned} 2a-b &= 2\sqrt{2n+1} - \sqrt{3n+1} \\ &= \frac{4(2n+1)-(3n+1)}{2\sqrt{2n+1}+\sqrt{3n+1}} \\ &= \frac{5n}{2\sqrt{2n+1}+\sqrt{3n+1}} \\ &\geq \frac{5n}{2\sqrt{3n}+\sqrt{4n}} \\ &= \frac{5}{2+2\sqrt{3}}\sqrt{n} \\ &> 1, \end{aligned}$$

where the last inequality is obtained using $n \geq 3$. This shows that $2a-b > 1$. From Eq. (1), it follows that $5n+3$ is a composite number. ■

Exercise 2 (RMO 2011b P3)**.** Let $a, b, c > 0$. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression, and if a^2+b^2, b^2+c^2, c^2+a^2 are in geometric progression, show that $a = b = c$.

Solution 2. The given conditions yield

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b},\quad \frac{b^2+c^2}{a^2+b^2}=\frac{c^2+a^2}{b^2+c^2}.$$

The first condition gives

$$\frac{a-b}{ab}=\frac{b-c}{bc}=\frac{a-c}{b(a+c)}. \tag{2}$$

On the contrary, let us assume that $a\neq c$. Using $a,c>0$, we get $a^2\neq c^2$, and hence, the second condition gives

$$\frac{b^2+c^2}{a^2+b^2}=\frac{c^2+a^2}{b^2+c^2}=\frac{a^2-b^2}{c^2-a^2}=\frac{a+b}{c+a}\frac{a-b}{c-a}=-\frac{a+b}{c+a}\frac{a}{a+c},$$

which is impossible since a,b,c are positive. This proves that $a=c$. Using Equation (2), it follows that $a=c$. This completes the proof. ■

Exercise 3 (RMO 2011b P4,). Find the number of 4-digit numbers with distinct digits chosen from the set $\{0,1,2,3,4,5\}$ in which no two adjacent digits are even.

Solution 3. The 4-digit numbers whose no two adjacent digits are even are of the form

$$eooo, eooe, eoeo, oeoo, oeoe, ooeo, oooe$$

where e (resp. o) denotes the evenness (resp. oddness) of the corresponding digit. The numbers of 4-digit numbers of these types with distinct digits chosen from $\{0,1,2,3,4,5\}$ and not containing two even numbers as adjacent digits are

$$2\cdot 3!, 2\cdot 3\cdot 2\cdot 2, 2\cdot 3\cdot 2\cdot 2, 3\cdot 3!, 3\cdot 2\cdot 3\cdot 2, 3\cdot 3!, 3\cdot 3!$$

respectively. Hence, there are

$$2\cdot 3!+2\cdot 3\cdot 2\cdot 2+2\cdot 3\cdot 2\cdot 2+3\cdot 3!+3\cdot 2\cdot 3\cdot 2+3\cdot 3!+3\cdot 3!=150$$

4-digit numbers satisfying the given conditions. ■

Exercise 4 (RMO 2011b P5,). Let $ABCD$ be a convex quadrilateral. Let E,F,G,H be the midpoints of AB,BC,CD,DA respectively. If AC,BD,EG,FH concur at a point O, prove that $ABCD$ is a parallelogram.

Walkthrough —

(a) Show that EH,FG are parallel and equal in length, hence $EFGH$ is a parallelogram.

(b) Let P be the point of intersection of AC and EH. Show that the diagonals AO and EH of the quadrilateral $AEOH$ bisect each other.

(c) Show that AE is parallel and equal to OH.

The content posted here and at this blog by Evan Chen are quite useful.

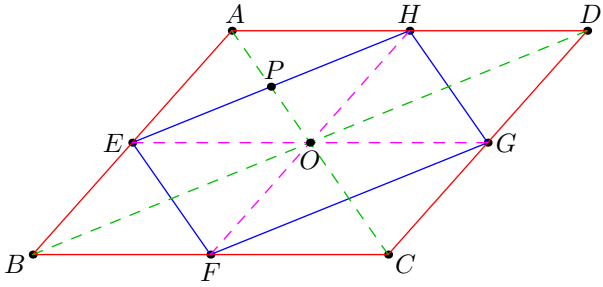


Figure 1: RMO 2011b P5, Exercise 4

(d) Show that AB is parallel and equal to FH.

(e) Similarly, show that CD is parallel and equal to FG.

(f) Conclude that $ABCD$ is a parallelogram.

Solution 4. Since E, H are the midpoints of AB, AD respectively, it follows that EH is parallel to BD and $EH = \frac{1}{2}BD$. Similarly, considering the midpoints F, G of BC, CD respectively, we get that FG is parallel to BD and $FG = \frac{1}{2}BD$. Thus, EH and FG are parallel and $EH = FG$. Hence, $EFGH$ is a parallelogram, and so, the diagonals EG, FH bisect each other.

Let P be the point of intersection of AC and EH. Since EP is parallel to BO and E is the midpoint of AB, it follows that $AP = PO$. Considering the midpoints H, G of AD, CD respectively, we get that HG is parallel to OP. Since EG, FH bisect each other at O, it follows that $EO = OG$. Combining this with the fact that HG is parallel to OP, we get that $EP = PH$. This shows that the diagonals AO, EH of the quadrilateral $AEOH$ bisect each other. This implies that the triangles $\triangle APE$ and $\triangle OPH$ are congruent. Hence, $AE = OH$ and AE is parallel to OH. Also, since E is the midpoint of AB and O is the midpoint of FH, it follows that AB is parallel to FH and $AB = FH$. Similarly, considering the midpoints F, G of BC, CD respectively, we get that CD is parallel to FG and $CD = FG$. Thus, $ABCD$ is a parallelogram. ■

Remark. The fact that the midpoints of the sides of any quadrilateral form a parallelogram, is known as Varignon's theorem.

Exercise 5 (RMO 2011b P6,). Find the largest real constant λ such that

$$\frac{\lambda abc}{a+b+c} \leq (a+b)^2 + (a+b+4c)^2 \tag{3}$$

for all positive real numbers a, b, c.

Solution 5. Let λ be a nonnegative real number. Then the above inequality holds for all $a, b, c > 0$ if and only if

$$\lambda x^2 c \leq (2x + c)(4x^2 + 4(x + 2c)^2)$$

holds for all $x, c > 0$. Noting that both the sides are homogeneous of degree three in x and y, it follows that the above inequality holds if and only if

$$\lambda x^2 \leq (2x + 1)(4x^2 + 4(x + 2)^2)$$

holds for all $x > 0$. Observe that

$$\begin{aligned}(2x + 1)(4x^2 + 4(x + 2)^2) &= 8(2x + 1)(x^2 + 2x + 2)\\ &= 8(2x^3 + 5x^2 + 6x + 2).\end{aligned}$$

This shows that for $x > 0$, the above inequality is equaivalent to

$$\frac{1}{2}\left(\frac{\lambda}{8} - 5\right) \leq x + \frac{3}{x} + \frac{1}{x^2}.$$

Let $a, b > 0$ be such that

$$x + \frac{3}{x} + \frac{1}{x^2} = ax + \frac{3}{x} + 2bx + \frac{1}{x^2}$$

holds for any $x > 0$, and $(3/a)^3 = (1/b)^2$ holds, or equivalently, a, b satisfy $a + 2b = 1$ and $(3/a)^3 = (1/b)^2$. Note that this holds for $a = 3/4, b = 1/8$. Observe that

$$x + \frac{3}{x} + \frac{1}{x^2} = \frac{3}{4}x + \frac{3}{x} + \frac{1}{8}x + \frac{1}{8}x + \frac{1}{x^2} \geq 3 + \frac{3}{4} = \frac{15}{4},$$

where equality holds if and only if $x = 2$. This proves that the largest real constant λ satisfying the given inequality for all $a, b, c > 0$ satisfies

$$\frac{1}{2}\left(\frac{\lambda}{8} - 5\right) = \frac{15}{4},$$

or equivalently, $\lambda = 100$ holds. ■