RMO 2011a

Exercise 1 (RMO 2011a P2,). Let $(a_1, a_2, a_3, \ldots, a_{2011})$ be a permutation of the numbers $1, 2, 3, \ldots, 2011$. Show that there exist two numbers j, k such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$.

Solution 1. Since $a_1, a_2, \ldots, a_{2011}$ is a permutation of $1, 2, \ldots, 2011$, the sum $\sum_{j=1}^{2011}(a_j - j)$ is equal to zero, which gives

$$\sum_{\substack{1\leq j\leq 2011\\ a_j - j > 0}} (a_j - j) = -\sum_{\substack{1\leq j\leq 2011\\ a_j - j < 0}} (a_j - j). \tag{1}$$

For any $1 \leq j \leq 2011$, note that $-2010 \leq a_j - j \leq 2010$, that is, $0 \leq |a_j - j| \leq 2010$ holds. If $|a_j - j|, |a_k - k|$ are equal for no j, k with $1 \leq j < k \leq 2011$, then $|a_1 - 1|, |a_2 - 2|, \ldots, |a_{2011} - 2011|$ are equal to $0, 1, 2, \ldots, 2010$ in some order, and hence

$$|a_1 - 1| + |a_2 - 2| + \cdots + |a_{2011} - 2011| = 1 + 2 + \cdots + 2010 = 1005 \cdot 2011,$$

which is odd and thus contradicts Eq. (1). So there exist two numbers j, k such that $1 \leq j < k \leq 2011$ and $|a_j - j| = |a_k - k|$. ■

Exercise 2 (RMO 2011a P3,). A natural number n is chosen strictly between two consecutive perfect squares. The smaller of these two squares is obtained by subtracting k from n and the larger by adding ℓ to n. Prove that $n - k\ell$ is a perfect square.

Solution 2. Let m be a positive integer such that

$$m^2 = n - k, (m+1)^2 = n + \ell$$

holds. This yields

$$\begin{aligned} n - k\ell &= n - (n - m^2)((m+1)^2 - n) \\ &= n + n^2 - n(m^2 + (m+1)^2) + m^2(m+1)^2 \end{aligned}$$

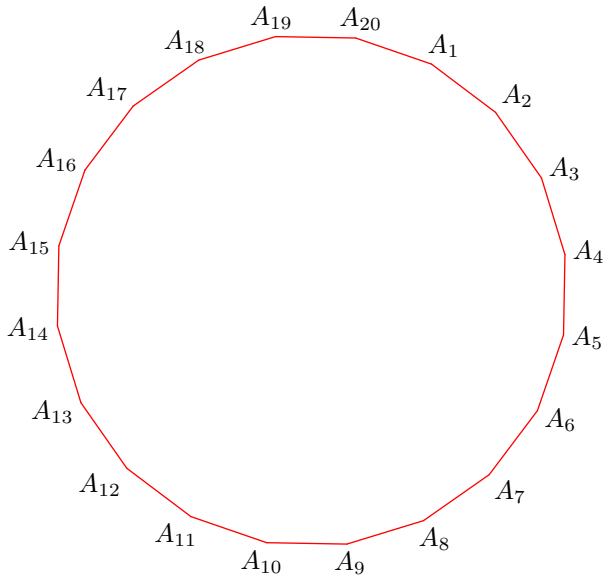


Figure 1: India RMO 2011, Exercise 3

$$= n^2 - n(2m^2 + 2m) + (m^2 + m)^2$$
$$= (n - m^2 - m)^2,$$

which is a perfect square. ■

Exercise 3 (RMO 2011a P4,). Consider a 20-sided convex polygon K, with vertices $A_1, A_2, \ldots, A_{20}$ in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not.)

Solution 3. In the following, by an admissible triple, we mean a three-element subset of the set of sides of K, satisfying the required condition. Thus, a triple is **not** ordered. Note that the pair (A_1A_2, A_4A_5) can be completed to the following admissible triples

$$(A_1A_2, A_4A_5, A_7A_8), (A_1A_2, A_4A_5, A_8A_9), \ldots, (A_1A_2, A_4A_5, A_{18}A_{19}),$$

and these are all the admissible triples containing the pair (A_1A_2, A_4A_5). Thus there are exactly $18 - 7 + 1 = 12$ admissible triples containing the pair (A_1A_2, A_4A_5). More generally, for any $4 \leq i \leq 15$, there are exactly $18 - (i + 3) + 1 = 16 - i$ admissible triples containing the pair (A_1A_2, A_iA_{i+1}).

Thus it follows that there are exactly

$$\sum_{i=4}^{15}(16-i) = 1+2+\cdots+12 = 78$$

admissible triples containing the side A_1A_2. So the required number of admissible triples is $\frac{78\cdot 20}{3} = 520$. ■

Exercise 4 (RMO 2011a P6,). Find all pairs (x, y) of real numbers such that

$$16^{x^2+y} + 16^{x+y^2} = 1.$$

Solution 4. By AM-GM inequality, we have

$$1 \geq 2\sqrt{16^{x^2+y}\cdot 16^{x+y^2}} = 2^{1+2(x^2+y+x+y^2)},$$

which implies that $x^2+y^2+x+y+\frac{1}{2} \leq 0$. This shows that $\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2 \leq 0$. Since x, y are real numbers, we get $x = y = -\frac{1}{2}$.

Note that the pair $(-\frac{1}{2}, -\frac{1}{2})$ satisfies the given equation. Hence, $(-\frac{1}{2}, -\frac{1}{2})$ is the only solution to the given equation. ■