RMO 2010

Exercise 1 (RMO 2010 P2,). Let

$$P_1(x) = ax^2 - bx - c, P_2(x) = bx^2 - cx - a, P_3(x) = cx^2 - ax - b$$

be three quadratic polynomials where a, b, c are nonzero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$.

Solution 1. Since $P_1(\alpha), P_2(\alpha)$ are equal, we get

$$(a-b)\alpha^2 = (b-c)\alpha + (c-a),$$

which gives

$$(a-b)(\alpha^2+1) = (b-c)(\alpha-1).$$

Similarly, using $P_2(\alpha) = P_3(\alpha)$, we obtain

$$(b-c)(\alpha^2+1) = (c-a)(\alpha-1).$$

If $\alpha = 1$, then it follows that $a = b = c$. Henceforth, let us assume that $\alpha \neq 1$. Then the above yields

$$(a-b)(c-a) = (b-c)^2.$$

Using a similar argument as above, it follows that

$$(b-c)(a-b) = (c-a)^2, (c-a)(b-c) = (a-b)^2.$$

Adding these equations, we obtain

$$a^2 + b^2 + c^2 = ab + bc + ca.$$

Since a, b, c are real, it follows that $a = b = c$. This completes the proof. ■

Exercise 2 (RMO 2010 P3,). Find the number of 4-digit numbers (in base 10) having non-zero digits and which are divisible by 4 but not by 8.

Solution 2. For any $1 \leq a, b, c \leq 9$, the four consecutive even integers

$$1000a + 100b + 10c + 2, 1000a + 100b + 10c + 4,$$

$$1000a + 100b + 10c + 6, 1000a + 100b + 10c + 8$$

are congruent to $0, 2, 4, 6 \pmod 8$ in some order. So exactly one of them (which is congruent to 4 modulo 8) is divisible by 4 but not by 8. Hence for any three integers $1 \leq a, b, c \leq 9$, there is a unique nonzero digit d (that is, $1 \leq d \leq 9$) such that $1000a + 100b + 10c + d$ is divisible by 4, but not by 8. So there are $9 \cdot 9 \cdot 9 = 729$ four-digit numbers (in base 10) with nonzero digits which are divisible by 4, but not by 8. ■

Exercise 3 (RMO 2010 P4,). Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer.

Solution 3. Let a, b, c be distinct positive integers with the least possible sum such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{p}{a} = \frac{q}{b} = \frac{r}{c} \tag{1}$$

holds, where p, q, r are positive integers. Since a, b, c are positive, it follows that p, q, r are greater than 1. By reordering a, b, c if necessary, we may and do assume that $a < b < c$, or equivalently, $p < q < r$ holds. Note that

$$\frac{3}{a} > \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{p}{a},$$

which gives $p = 2$. This yields

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c} < \frac{2}{b},$$

which implies that $q < 2p = 4$, and using $p < q$, we obtain $q = 3$. Now Eq. (1) gives

$$2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{p}{a} + \frac{q}{b},$$

and hence, we obtain $\frac{2}{c} = \frac{1}{b}$, which yields $r = 2q = 6$. This shows that $a : b : c$ is equal to $2 : 3 : 6$. Noting that $2, 3, 6$ have no common divisor larger than 1, it follows that $a + b + c$ is a multiple of $2 + 3 + 6 = 11$. Note that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1,$$

which is an integral multiple of the reciprocal of any of $2, 3, 6$. This proves that $2, 3, 6$ are three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer. Moreover, this proof also shows that these are unique up to reordering. ■

Exercise 4 (RMO 2010 P6,). For each integer $n \geq 1$, define

$$a_n = \left[\frac{n}{[\sqrt{n}]}\right],$$

where $[x]$ denotes the largest integer not exceeding x, for any real number x. Find the number of all n in the set $\{1, 2, 3, \ldots, 2010\}$ for which $a_n > a_{n+1}$.

Solution 4. Let n be a positive integer such that $a_n > a_{n+1}$ holds. Note that $[\sqrt{n}], [\sqrt{n+1}]$ are not equal, otherwise, we would obtain

$$\frac{n+1}{[\sqrt{n+1}]} = \frac{n+1}{[\sqrt{n}]} \geq \frac{n}{[\sqrt{n}]},$$

implying $a_{n+1} \geq a_n$. Let k denote the largest integer satisfying $k^2 \leq n$. In other words, k denotes the integer $[\sqrt{n}]$. Since $[\sqrt{n}], [\sqrt{n+1}]$ are not equal, we obtain $[\sqrt{n+1}] > k$, which gives $[\sqrt{n+1}] \geq k+1$. This shows that $n+1 \geq (k+1)^2$. Since k is the largest integer satisfying $k^2 \leq n$, we obtain $n+1 = (k+1)^2$. We conclude that if $a_n > a_{n+1}$ holds, then $n+1$ is a perfect square. Also note that the converse of this statement holds, that is, if $n+1$ is a perfect square, then $a_n > a_{n+1}$ holds. Indeed, if $n+1 = m^2$ holds for some positive integer m, then it follows that

$$\frac{n}{[\sqrt{n}]} - \frac{n+1}{[\sqrt{n+1}]} = \frac{m^2-1}{m-1} - \frac{m^2}{m} = 1,$$

which yields

$$\left[\frac{n}{[\sqrt{n}]}\right] \geq \left[\frac{n+1}{[\sqrt{n+1}]}\right] \geq 1 > 0.$$

This proves that for a positive integer n, the inequality $a_n > a_{n+1}$ holds if and only if $n+1$ is a perfect square. Noting that

$$45^2 - 1 = 2025 - 1 > 2010, 44^2 = 2025 + 1 - 90 < 2010$$

holds, we conclude that the positive integers n in the set $\{1, 2, \ldots, 2010\}$ satisfying $a_n > a_{n+1}$ are precisely

$$2^2 - 1, 3^2 - 1, \ldots, 44^2 - 1.$$

So there are 43 values of n satisfying the given condition. ■