# RMO 2009

**Exercise 1** (RMO 2009 P2, ). Show that there is no integer $a$ such that $a^2 - 3a - 19$ is divisible by 289.

**Solution 1.** If $a^2 - 3a - 19$ is divisible by 289 for some integer $a$, then

$$4(a^2 - 3a - 19) = (2a)^2 - 12a - 76 = (2a - 3)^2 - 85$$

would be divisible by $289 = 17^2$, and hence $2a - 3$ would be a multiple of 17, and consequently, $17^2$ would divide 85, which is impossible. This proves the result. ■

**Exercise 2** (RMO 2009 P3, ). Show that $3^{2008} + 4^{2009}$ can be written as product of two positive integers each of which is larger than $2009^{182}$.

**Solution 2.** Note that $3^{2008}+4^{2009}$ is equal to $4x^4+y^4$ where $x = 4^{502}, y = 3^{502}$. Also note that

$$3^{2008} + 4^{2009} = (2x^2 + y^2 - 2xy)(2x^2 + y^2 + 2xy)$$

holds. Since $x \neq y$, it follows that the integers $2x^2+y^2+2xy, 2x^2+y^2-2xy$ are strictly larger than $x^2$. So it suffices to show $x^2 \geq 2009^{182}$, which is equivalent to $x \geq 2009^{91}$, which follows since

$$x = 4^{502} = 2^{1004} > 2^{11 \cdot 91} = 2048^{91} > 2009^{91}$$

holds. ■

**Exercise 3** (RMO 2009 P4, ). Find the sum of all 3-digit natural numbers which contain at least one odd digit and at least one even digit.

**Walkthrough** —

**(a)** Note that the 3-digit natural numbers containing at least one odd digit, and at least one even digit, form the set $S \setminus (E \cup O)$, where

$S =$ the set of 3-digit integers,

$E =$ the set of 3-digit numbers having even digits only,

$O =$ the set of 3-digit numbers having odd digits only.

**(b)** Apply the inclusion-exclusion principle.

**Solution 3.** Let $E$ (resp. $O$) denote the set of 3-digit numbers having even (resp. odd) digits only. Denote by $S$ the set of 3-digit integers. Note that the 3-digit numbers with at least one even digit and at least one odd digit form the set $S \setminus (E \cup O)$. By the inclusion-exclusion principle, the size of the set $S \setminus (E \cup O)$ is equal to

$$\sum_{s \in S} s - \sum_{x \in E} x - \sum_{y \in O} y.$$

Note that

$$\sum_{s \in S} s = 10 + 101 + \cdots + 999 = 1099 \cdot 450.$$

Observe that there are $5^3$ elements on $O$. Moreover, for any $1 \leq k \leq 3$ and for any $d \in \{1, 3, 5, 7, 9\}$, there are $5^2$ elements of $O$ that contain $d$ in the $k$-th digit. It follows that

$$\begin{aligned}\sum_{y \in O} y &= 100 \cdot 5^2(1+3+5+7+9) + 10 \cdot 5^2(1+3+5+7+9) \\ &\quad + 5^2(1+3+5+7+9) \\ &= 111 \cdot 25 \cdot 25.\end{aligned}$$

Similarly, there are $5^3$ integers of the form $100a + 10b + c$ with $a, b, c \in \{0, 2, 4, 6, 8\}$, and their sum is equal to

$$\begin{aligned}&100 \cdot 5^2(0+2+4+6+8) + 10 \cdot 5^2(0+2+4+6+8) + 5^2(0+2+4+6+8) \\ &= 111 \cdot 25 \cdot 20.\end{aligned}$$

Note that there are $5^2$ integers of the form $10b + c$ with $b, c \in \{0, 2, 4, 6, 8\}$, and their sum is equal to

$$10 \cdot 5(0+2+4+6+8) + 5(0+2+4+6+8) = 11 \cdot 5 \cdot 20.$$

This shows that

$$\sum_{x \in E} x = 111 \cdot 25 \cdot 20 - 11 \cdot 5 \cdot 20 = 54400.$$

It follows that

$$\begin{aligned}\sum_{s \in S} s - \sum_{x \in E} x - \sum_{y \in O} y &= 1099 \cdot 450 - 111 \cdot 25 \cdot 25 - 54400 \\ &= 495000 - 450 - 62500 - 6250 - 625 - 54400 \\ &= 495000 - 450 - 123775 \\ &= 370775.\end{aligned}$$

■

**Exercise 4** (RMO 2009 P5, ). A convex polygon is such that the distance between any two vertices does not exceed 1.

(i) Prove that the distance between any two points on the boundary of the polygon does not exceed 1.

(ii) If $X$ and $Y$ are two distinct points inside the polygon, prove that there exists a point $Z$ on the boundary of the polygon such that $XZ+YZ \leq 1$.

**Solution 4.** Note that for any four complex numbers $z_1, z_2, z_3, z_4$ and any $0 \leq t, s \leq 1$, we have

$$\begin{aligned}&(tz_1+(1-t)z_2)-(sz_3+(1-s)z_4)\\&=t(z_1-z_3)+(t-s)z_3+(1-s)(z_2-z_4)+(s-t)z_2\\&=t(z_1-z_3)+(s-t)(z_3-z_2)+(1-s)(z_2-z_4)\end{aligned}$$

and

$$\begin{aligned}&(tz_1+(1-t)z_2)-(sz_3+(1-s)z_4)\\&=s(z_1-z_3)+(t-s)z_1+(1-t)(z_2-z_4)+(s-t)z_4\\&=s(z_1-z_3)+(t-s)(z_1-z_4)+(1-t)(z_2-z_4).\end{aligned}$$

If the distance between no two of $z_1, z_2, z_3, z_4$ exceeds 1, then

$$|(tz_1+(1-t)z_2)-(sz_3+(1-s)z_4)| \leq \begin{cases} t+(s-t)+(1-s)=1 & \text{if } s \geq t,\\ s+(t-s)+(1-t)=1 & \text{if } t \geq s.\end{cases}$$

This proves part (i).

Extending $XY$ to the boundary of the polygon, we find two points $Z_1, Z_2$ on the boundary such that $Z_1, X, Y, Z_2$ are collinear. Note that

$$\begin{aligned}(XZ_1+YZ_1)+(XZ_2+YZ_2)&=(XZ_1+XZ_2)+(YZ_1+YZ_2)\\&=2Z_1Z_2\\&\leq 2,\end{aligned}$$

which shows that $XZ_1+YZ_1 \leq 1$ or $XZ_2+YZ_2 \leq 1$ holds. This proves part (ii). ■

**Exercise 5** (RMO 2009 P6, ). In a book with page numbers from 1 to 100 some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?

**Solution 5.** Suppose $r$ pages are torn off. Denote the page numbers of the torn pages by $2n_1-1, 2n_1, 2n_2-1, 2n_2, \ldots, 2n_r-1, 2n_r$. So

$$1+2+\cdots+50$$

$$= 4949 + 2n_1 - 1 + 2n_1 + 2n_2 - 1 + 2n_2 + \cdots + 2n_r - 1 + 2n_r,$$

which gives

$$4(n_1 + \cdots + n_r) - r = 5050 - 4949 = 101. \tag{1}$$

Consequently, we obtain

$$\begin{aligned} 101 &= 4(n_1 + \cdots + n_r) - r \\ &\geq 4(1 + 2 + \cdots + r) - r \\ &= 2r(r+1) - r \\ &= r(2r+1), \end{aligned}$$

and hence

$$r \leq \frac{-1 + \sqrt{1 + 4 \cdot 2 \cdot 101}}{4} = \frac{-1 + \sqrt{809}}{4} \leq 7.$$

Moreover, Equation (1) implies that $r \equiv 3 \pmod 4$. Consequently, three pages are torn. ■



The content posted here and at this blog by Evan Chen are quite useful.