RMO 2008

Exercise 1 (RMO 2008 P2,). Prove that there exist two infinite sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ of positive integers such that the following conditions hold simultaneously:

(i) $0 < a_1 < a_2 < a_3 < \cdots$,

(ii) $a_n < b_n < a_n^2$ for all $n \geq 1$,

(iii) $a_n - 1$ divides $b_n - 1$ for all $n \geq 1$,

(iv) $a_n^2 - 1$ divides $b_n^2 - 1$ for all $n \geq 1$.

Solution 1. We claim that it suffices to prove that for any positive integer N, there exist integers $a, b > N$ such that $a < b < a^2$, and $a - 1$ divides $b - 1$, $a^2 - 1$ divides $b^2 - 1$. Indeed, if this statement is true, then there exist positive integers a_1, b_1 such that $a_1 < b_1 < a_1^2$, and $a_1 - 1$ divides $b_1 - 1$, $a_1^2 - 1$ divides $b_1^2 - 1$. Moreover, if for some positive integer $n \geq 1$, there are positive integers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ such that $a_1 < a_2 < \cdots < a_n, b_1 < b_2 < \cdots < b_n$, and $a_k - 1$ divides $b_k - 1$ and $a_k^2 - 1$ divides $b_k^2 - 1$ for any $1 \leq k \leq n$, then by the above statement, there exist integers a, b such that $a, b > a_n + b_n$, $a < b < a^2$, and $a - 1$ divides $b - 1$, $a^2 - 1$ divides $b^2 - 1$, then one can define $a_{n+1} = a, b_{n+1} = b$. Applying induction, we obtain two infinite sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ as desired. Now it remains to prove that for any positive integer N, there exist integers $a, b > N$ such that $a < b < a^2$, and $a - 1$ divides $b - 1$, $a^2 - 1$ divides $b^2 - 1$.

Note that if a, b are two integers such that $a - 1$ divides $b - 1$, then $b = 1 + (a - 1)k$ for some integer k, and hence

$$\begin{aligned} b^2 - 1 &= (b-1)(b+1) \\ &= k(a-1)((a-1)k + 2) \\ &= k(a-1)((a+1)k - (2k-2)) \\ &= k^2(a^2-1) - 2k(k-1)(a-1) \end{aligned}$$

holds, which shows that b^2-1 is divisible by a^2-1 if $a+1$ divides $2k$, which holds if a is odd and $k=\frac{a+1}{2}$. In fact, for any positive integer N, setting

$$a=2N+1, \quad b=1+(a-1)\frac{a+1}{2}=\frac{a^2+1}{2}=2N^2+2N+1,$$

the inequality $a<b<a^2$ follows, and $a-1=2N$ divides $b-1=2N(N+1)$, $a^2-1=4N(N+1)$ divides $b^2-1=4N(N+1)(N^2+N+1)$. This completes the proof. ■

> **Remark.** The above argument (that is, **only after having** the above argument, it) shows that any strictly increasing sequence $\{a_n\}_{n\geq 1}$ of odd positive integers with $a_1\geq 3$, and the $\{b_n\}_{n\geq 1}$, defined by
>
> $$b_n=\frac{a_n^2+1}{2} \quad \text{for any integer } n\geq 1,$$
>
> have the required properties.

Exercise 2 (RMO 2008 P3,). Suppose a and b are real numbers such that the roots of the cubic equation ax^3-x^2+bx-1 are positive real numbers. Prove that

(i) $0<3ab\leq 1$,

(ii) $b\geq\sqrt{3}$.

Solution 2. Let α,β,γ denote the roots of the polynomial ax^3-x^2+bx-1. Note that

$$\begin{aligned}\alpha+\beta+\gamma&=\frac{1}{a},\\ \alpha\beta+\beta\gamma+\gamma\alpha&=\frac{b}{a},\\ \alpha\beta\gamma&=\frac{1}{a}\end{aligned}$$

holds. Since α,β,γ are positive, it follows that a,b are positive. Using

$$(\alpha+\beta+\gamma)^2\geq 3(\alpha\beta+\beta\gamma+\gamma\alpha),$$

we obtain $\frac{1}{a^2}\geq\frac{3b}{a}$. Since a is positive, it follows that $3ab\leq 1$. Note that

$$\begin{aligned}(\alpha\beta+\beta\gamma+\gamma\alpha)^2&=\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2\alpha\beta\gamma(\alpha+\beta+\gamma)\\ &\geq\frac{(\alpha\beta+\beta\gamma+\gamma\alpha)^2}{3}+2\alpha\beta\gamma(\alpha+\beta+\gamma)\end{aligned}$$

holds, we get $\frac{b^2}{a^2}\geq\frac{b^2}{3a^2}+2\frac{1}{a^2}$, which gives $b^2\geq 3$. Since b is positive, we conclude $b\geq\sqrt{3}$. ■

The content posted here and at this blog by Evan Chen are quite useful.

Exercise 3 (RMO 2008 P4,). Find the number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits $0, 1, 2, 3$ occurs at least once in them.

Solution 3. If $0, 1, 2, 3, a, b$ denotes the digits of a 6-digit number such that the sum of their digits is 10, then $a + b = 4$, *i.e.*, $\{a, b\}$ is equal to $\{0, 4\}, \{1, 3\}$ or $\{2, 2\}$. So the digits of the 6-digit numbers satisfying the given conditions are $0, 0, 1, 2, 3, 4$, or $0, 1, 1, 2, 3, 3$, or $0, 1, 2, 2, 2, 3$. So the number of the 6-digit numbers satisfying the given conditions is

$$\left(\frac{6!}{2!} - 5!\right) + \left(\frac{6!}{2!2!} - \frac{5!}{2!2!}\right) + \left(\frac{6!}{3!} - \frac{5!}{3!}\right) = 240 + 150 + 100 = 490.$$

■

Exercise 4 (RMO 2008 P5,). Three nonzero real numbers a, b, c are said to be in harmonic progression if $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$. Find all three term harmonic progressions a, b, c of strictly increasing positive integers in which $a = 20$ and b divides c.

Solution 4. Let b, c be positive integers such that $20 < b < c$ holds, and b divides c and

$$\frac{1}{20} + \frac{1}{c} = \frac{2}{b}$$

holds. For some positive integer $k > 1$, we have $c = bk$, which yields

$$\frac{1}{20} + \frac{1}{bk} = \frac{2}{b},$$

and this gives

$$bk = 20(2k - 1).$$

Since $k, 2k - 1$ are relatively prime and b is an integer, it follows that k divides 20, and hence, k is equal to one of $2, 4, 5, 10$ or 20. This shows that (a, b, c) is equal to one of

$$(20, 30, 60), (20, 35, 140), (20, 36, 180), (20, 38, 380), (20, 39, 780).$$

Note that any of the above triples satisfy the required conditions. This proves that the above ones are all the three term harmonic progressions satisfying the required conditions. ■