RMO 2007

Exercise 1 (RMO 2007 P2,). Let a, b, c be three natural numbers such that $a < b < c$ and $\gcd(c-a, c-b) = 1$. Suppose there exists an integer d such that $a+d, b+d, c+d$ form the sides of a right-angled triangle. Prove that there exist integers ℓ, m such that $c+d = \ell^2 + m^2$.

Solution 1. Since $a < b < c$, and the integers $a+d, b+d, c+d$ form the sides of a right-angled triangle, it follows that

$$(c+d)^2 = (a+d)^2 + (b+d)^2. \tag{1}$$

Writing the above as a quadratic equation d, we obtain

$$d^2 + 2(a+b-c)d + a^2 + b^2 - c^2 = 0,$$

which implies that

$$d = (c-a-b) \pm \sqrt{(a+b-c)^2 - (a^2+b^2-c^2)}.$$

Since $a+b+d-c = (a+d)+(b+d)-(c+d)$ is positive, we obtain

$$\begin{aligned} d &= (c-a-b) + \sqrt{(a+b-c)^2 - (a^2+b^2-c^2)} \\ &= (c-a-b) + \sqrt{2c^2 + 2ab - 2bc - 2ca} \\ &= (c-a-b) + \sqrt{2(c-a)(c-b)}. \end{aligned}$$

Since d is an integer, it follows that $2(c-a)(c-b)$ is a perfect square. Note that the integers $c-a, c-b$ are positive and relatively prime. So there are positive integers m, n such that $c-a, c-b$ are equal to $2m^2, n^2$ in some order. This yields

$$\begin{aligned} c+d &= (c-a) + (c-b) + \sqrt{2(c-a)(c-b)} \\ &= 2m^2 + n^2 + 2mn \\ &= m^2 + (m+n)^2, \end{aligned}$$

which proves the result. ■

Remark. After obtaining Eq. (1), one may also argue as follows. Note that Eq. (1) implies that a divisor of any two of $a+d, b+d, c+d$, divides all of them, and hence also divides the differences $c-a, c-b$, which are coprime. This shows that $a+d, b+d, c+d$ are pairwise coprime integers, and $(a+d, b+d, c+d)$ is a primitive Pythagorean triple. It follows (**how?**) that $c+d$ is equal to ℓ^2+m^2 for some integers ℓ, m.

It is **not a good idea** to apply the classification of primitive Pythagorean triples to conclude that $c+d$ is the sum of two squares, since the above solution of Exercise 1 is **no different from** (one proof of) the classification of primitive Pythagorean triples, which is provided below.

A careful reading of the above solution of Exercise 1 leads to the following proof of the primitive Pythagorean triples as follows.

Lemma 1 (Classification of primitive Pythagorean triples)
Let $1 \le x \le y \le z$ be integers satisfying

$$x^2 + y^2 = z^2.$$

Then the following statements are equivalent.

(i) Some two of x, y, z are relatively prime.

(ii) Any two of x, y, z are relatively prime.

(iii) The integers $z-x, z-y$ are relatively prime.

If any of the above conditions holds, then there are relatively prime positive integers $a > b$ such that x, y are equal to $a^2-b^2, 2ab$ in some order, and z is equal to a^2+b^2.

For a **geometric proof** of the above, we refer to [**ST15**, §1.1].

Proof. Note that

$$x^2 = (z-y)(z+y), \quad y^2 = (z-x)(z+x)$$

holds. So any common prime divisor of $z-x, z-y$ is also a common divisor of x, y. Using $x^2+y^2=z^2$, it follows that any common divisor of x, y is also a common divisor of $z-x, z-y$. The equivalence of the three statements follows.

Assume that one of the given conditions holds. Note that

$$\begin{aligned}(x+y-z)^2 &= (x+y-z)^2 - (x^2+y^2-z^2) \\ &= 2z^2 + 2xy - 2z(x+y) \\ &= 2(z-x)(z-y)\end{aligned}$$

The content posted here and at this blog by Evan Chen are quite useful.

holds. Since $z-x, z-y$ are relatively prime, and $2(z-x)(z-y)$ is a perfect square, it follows that there are positive integers m, n such that $z-x, z-y$ are equal to $2m^2, n^2$ in some order. Note that

$$\begin{aligned} z &= (z-x)+(z-y)+(x+y-z) \\ &= 2m^2+n^2+2mn \\ &= m^2+(m+n)^2, \end{aligned}$$

and this implies that

$$\begin{aligned} z-2m^2 &= (m+n)^2-n^2, \\ z-n^2 &= 2m(m+n). \end{aligned}$$

Putting

$$a = m+n, b = m,$$

it follows that x, y are equal to $a^2-b^2, 2ab$ in some order, and z is equal to a^2+b^2. Since x, y are relatively prime, we get that the integers a, b are relatively prime. □

Exercise 2 (RMO 2007 P3,). Find all pairs (a, b) of real numbers such that whenever α is a root of $x^2+ax+b=0$, α^2-2 is also a root of the equation.

Solution 2. Let a, b be real numbers such that for any root α of $x^2+ax+b=0$, α^2-2 is also a root. Denote the roots of x^2+ax+b by α, β. There are the following possibilities.

(1) $\alpha^2-2=\alpha$, $\beta^2-2=\beta$,

(2) $\alpha^2-2=\beta$, $\beta^2-2=\alpha$,

(3) $\alpha^2-2=\beta^2-2=\alpha$,

(4) $\alpha^2-2=\beta^2-2=\beta$.

If $\alpha=\beta$, then these four cases are equivalent to

$$\alpha^2-2=\beta^2-2=\alpha=\beta,$$

which shows that α is equal to 2 or -1, and hence (a, b) is equal to $(-4, 4)$ or $(2, 1)$.

It remains to consider the case that $\alpha \neq \beta$, which we assume from now on.

In Case (1), α, β satisfy the equation $X^2-X-2=0$. So (α, β) is equal to $(2, -1)$ or $(-1, 2)$, and hence (a, b) is equal to $(-1, -2)$.

In Case (2), we have $\alpha^2-\beta^2=\beta-\alpha$, which gives $\alpha+\beta=-1$ (since $\alpha \neq \beta$). So

$$\alpha\beta = \frac{1}{2}(\alpha+\beta)^2-\frac{1}{2}(\alpha^2+\beta^2) = \frac{1}{2}(\alpha+\beta)^2-\frac{1}{2}(\alpha+\beta+4) = \frac{1}{2}-\frac{3}{2} = -1.$$

This shows that α, β are roots of the quadratic polynomial $x^2 + x - 1$, and hence, (a, b) is equal to $(1, -1)$.

In Case (3), note that α is equal to 2 or -1. Using $\beta^2 = 2 + \alpha$ and $\alpha \neq \beta$, it follows that (α, β) is equal to $(2, -2)$ or $(-1, 1)$, and hence (a, b) is equal to $(0, -4)$ or $(0, -1)$.

Similarly, in Case (4), (α, β) is equal to $(-2, 2)$ or $(-1, 1)$, which shows (a, b) is equal to $(0, -4)$ or $(0, -1)$.

So (a, b) is equal to one of $(-4, 4), (2, 1), (-1, -2), (1, -1), (0, -4), (0, -1)$.

Moreover, if (a, b) is equal to any of these six pairs, then it can be checked that for any root α of $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root.

We conclude that all the required pairs are $(-4, 4)$, $(2, 1)$, $(-1, -2)$, $(1, -1)$, $(0, -4)$, $(0, -1)$. ■

Exercise 3 (RMO 2007 P4, ). How many 6-digit numbers are there such that

(a) the digits of each number are all from the set $\{1, 2, 3, 4, 5\}$,

(b) any digit that appears in the number appears at least twice?

For example, 225252 is valid while 222133 is not.

Solution 3. Such a 6-digit number is formed

1. using a, a, a, a, a, a for some $a \in \{1, 2, 3, 4, 5\}$, or
2. using a, a, a, a, b, b or a, a, b, b, b, b for some $a, b \in \{1, 2, 3, 4, 5\}$ with $a \neq b$, or
3. using a, a, a, b, b, b for some $a, b \in \{1, 2, 3, 4, 5\}$ with $a \neq b$, or
4. using a, a, b, b, c, c for three distinct elements $a, b, c \in \{1, 2, 3, 4, 5\}$.

So there are

$$\begin{aligned} &5 + \binom{5}{2}\left(\frac{6!}{2!4!} + \frac{6!}{2!4!}\right) + \binom{5}{2}\frac{6!}{3!3!} + \binom{5}{3}\frac{6!}{2!2!2!} \\ &= 5 + \binom{5}{2}(15 + 15 + 20 + 90) \\ &= 1405 \end{aligned}$$

many 6-digit numbers satisfying the given conditions. ■

Exercise 4 (RMO 2007 P5, ). A trapezium $ABCD$, in which AB is parallel to CD, is inscribed in a circle with centre O. Suppose the diagonals AC and BD of the trapezium intersect at M, and $OM = 2$.

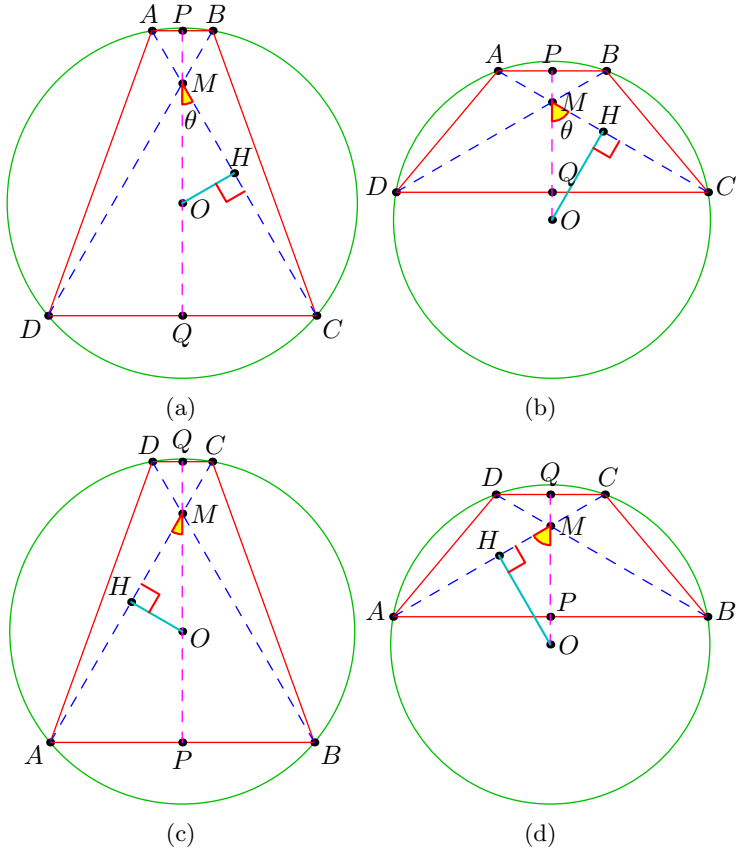


Figure 1: RMO 2007 P5, Exercise 4

1. If $\angle AMB$ is 60°, find, with proof, the difference between the lengths of the parallel sides.
2. If $\angle AMD$ is 60°, find, with proof, the difference between the lengths of the parallel sides.

Solution 4. Let P denote the intersection of AB with extended OM, and let Q denote the intersection of CD with OM extended. Note that $\angle BAC$ and $\angle BDC$ are equal, being angles subtended by the same chord BC. Since AB is parallel to CD, we have

$$\angle ABD = \angle BDC.$$

This shows that $\angle BAM, \angle ABM$ are equal, and hence $AM = BM$. Using $AM = BM$ and $OA = OB$, it follows that $\triangle AMO, \triangle BMO$ are congruent, and hence, $\angle AMO, \angle BMO$ are equal. This implies that $\angle AMP, \angle BMP$ are equal. It follows that $\triangle AMP, \triangle BMP$ are congruent. Thus, PQ bisects AB, and is perpendicular to it. Similarly, it follows that PQ bisects CD, and is perpendicular to it.

Note that $\triangle APM, \triangle CQM$ are similar, which gives

$$\frac{CQ}{CM} = \frac{AP}{AM}.$$

We obtain

$$|CD - AB| = 2|CM - AM|\frac{CQ}{CM}.$$

Let H denote the foot of the perpendicular from O to AC. Note that

$$CM = CH \pm HM, MA = HA \mp HM,$$

which yields

$$|CM - MA| = 2HM = 2OM \times \frac{HM}{OM}.$$

This gives

$$CD - AB = 4 \cdot OM \cdot \frac{HM}{OM} \cdot \frac{CQ}{CM}.$$

If $\angle AMB = 60^\circ$, then $\angle CMQ = 30^\circ$, which gives

$$\frac{HM}{OM} = \frac{\sqrt{3}}{2}, \quad \frac{CQ}{CM} = \frac{1}{2},$$

and hence,

$$CD - AB = 4 \cdot 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = 2\sqrt{3}.$$

Similarly, if $\angle AMD = 120^\circ$, then $\angle CMQ = 60^\circ$, which gives

$$\frac{HM}{OM} = \frac{1}{2}, \quad \frac{CQ}{CM} = \frac{\sqrt{3}}{2},$$

implying that

$$CD - AB = 4 \cdot 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}.$$

■

Exercise 5 (RMO 2007 P6,). Prove that

(a) $5 < \sqrt{5} + \sqrt[3]{5} + \sqrt[4]{5}$,

(b) $8 > \sqrt{8} + \sqrt[3]{8} + \sqrt[4]{8}$,

(c) $n > \sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n}$ for all integers $n \geq 9$.

Solution 5. Note that

$$\sqrt{5} > 2, \sqrt[3]{5} > \frac{8}{5}, \sqrt[4]{5} > \frac{7}{5}$$

holds, which gives the first inequality. Using $3 > \sqrt{8}, 3 > \sqrt[4]{8}$, we obtain the second inequality. Note that

$$\sqrt{n} + \sqrt[3]{n} + \sqrt[4]{n} < \sqrt{n} + \sqrt{n} + \sqrt{n} = 3\sqrt{n}$$

holds for any integer $n > 1$. Since $n \geq 3\sqrt{n}$ holds for any $n \geq 9$, the third inequality follows. ■

Remark. One may also use the AM-GM inequality to obtain the first inequality.

References

[**ST15**] Joseph H. Silverman and John T. Tate. *Rational points on elliptic curves*. Second. Undergraduate Texts in Mathematics. Springer, Cham, 2015, pp. xxii+332. ISBN: 978-3-319-18587-3; 978-3-319-18588-0. DOI: 10.1007/978-3-319-18588-0. URL: http://dx.doi.org/10.1007/978-3-319-18588-0 (cited p. 2)