RMO 2006

Exercise 1 (RMO 2006 P2,). Find the least possible value of $a+b$, where a, b are positive integers such that 11 divides $a+13b$ and 13 divides $a+11b$.

Solution 1. Let a, b be positive integers such that 11 divides $a+13b$ and 13 divides $a+11b$. It follows that 11 divides $6a+78b = 6a+b+77b$, and 13 divides $6a+66b = 6a+b+65b$. This shows that $6a+b$ is divisible by 11 and 13. Since the integers $11, 13$ are relatively prime, we obtain that $11 \cdot 13 = 143$ divides $6a+b$.

If $b \geq 4$, then we obtain

$$6a+6b = 6a+b+5b \geq 143+20 = 163,$$

which implies that

$$a+b \geq 28.$$

If $b = 1$, then $6a+b = 143$ admits no solutions in the integers, and hence $6a+b \geq 3 \cdot 143$ holds, which implies that $a+b \geq (6a+b)/6 > 28$. If $b = 2$, then $6a+b = 143$ admits no solutions in the integers, and hence $6a+b \geq 2 \cdot 143$ holds, which shows that $a+b \geq (6a+b)/6 > 28$. If $b = 3$, then $6a+b = 143$ admits no solutions in the integers, and hence $6a+b \geq 3 \cdot 143$ holds, which implies that $a+b \geq (6a+b)/6 > 28$. Considering these cases, it follows that $a+b \geq 28$.

Note that $a = 23, b = 5$ satisfy the given conditions. This shows that least possible value of $a+b$ accross all positive integers a, b satisfying the given divisibility conditions is 28. ■

Solution 2. Let a, b be positive integers such that 11 divides $a+13b$, 13 divides $a+11b$. It follows that $a+2b$ is equal to $11m$, and $a-2b$ is equal to $13n$ for some integers m, n with $m \geq 1$. This gives

$$a = \frac{11m+13n}{2}, b = \frac{11m-13n}{4}, a+b = \frac{33m+13n}{4}.$$

Since b is an integer, we obtain $3m \equiv n \bmod 4$. If (m, n) is equal to $(3, 1)$, then note that

$$A := \frac{11m+13n}{2}, B := \frac{11m-13n}{4}$$

are positive integers satisfying

$$A + B = 23 + 5 = 28,$$

and 11 divides $A + 2B$ and 13 divides $A - 2B$, or equivalently, 11 divides $A + 13B$ and 13 divides $A + 11B$. This shows that $a + b \leq 28$. Since m is positive, we get

$$\frac{33 + 13n}{4} \leq \frac{33m + 13n}{4} = a + b \leq 28,$$

$$\frac{11 - 13n}{4} \leq \frac{11m - 13n}{4} = b < a + b \leq 28,$$

which gives $-7 \leq n \leq 6$. Note that if n is nonnegative, then we obtain

$$\frac{33m}{4} \leq a + b \leq 28,$$

which gives $m \leq 3$, and when n is negative, we have

$$\frac{33m + 13n}{4} \leq 28,$$

which gives

$$33m \leq 112 - 13n \leq 112 + 13 \cdot 7 = 203,$$

implying $m \leq 6$. Moreover, we have

$$m \geq \frac{13}{11}|n| > |n|.$$

Noting that $m + n \equiv 0 \bmod 4$ holds, it follows that the pair (m, n) is equal to one of $(3, 1), (6, -2), (5, -1)$. If (m, n) is equal to $(6, -2), (5, -1)$, then $b = \frac{11m-13n}{4}$ is larger than 28. Consequently, (m, n) is equal to $(3, 1)$. This shows that the least value of $a + b$ is 28. ■

Exercise 2 (RMO 2006 P3,). If a, b, c are three positive real numbers, prove that

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq 3.$$

Solution 3. Note that

$$\frac{a^2 + 1}{b + c} + \frac{b^2 + 1}{c + a} + \frac{c^2 + 1}{a + b} \geq \frac{2a}{b + c} + \frac{2b}{c + a} + \frac{2c}{a + b} \geq 3,$$

where the first inequality follows from AM-GM inequality and the second inequality follows from Nesbitt's inequality **??**. ■

Exercise 3 (RMO 2006 P4,). A 6×6 square is dissected into 9 rectangles by lines parallel to its sides such that all these rectangles have only integer sides. Prove that there are always two congruent rectangles.

Solution 4. On the contrary, let us assume that it is possible to divide a 6×6 square into 9 disjoint subrectangles with each side having integer length, and parallel to one of the sides of the square such that no two subrectangles are congruent. Denote the areas of these subrectanlges by

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8 \leq a_9.$$

Note that if for some $1 \leq i \leq 9$, the integer a_i is a prime or equal to 1, then a_i is the area of exactly one of these 9 subrectangles. Noting that

$$4 = 1 \times 4 = 2 \times 2, \quad 6 = 1 \times 6, 2 \times 3$$

are the only factorizations of 4 and 6 over the positive integers upto reordering, it follows that if for some $1 \leq i \leq 9$, the integer a_i is equal to one of $4, 6, 8$, then it can be the area of at most two such subrectangles. Moreover, note that for no integer $1 \leq i \leq 9$, a_i is equal to 7 since these subrectangles are contained in a 6×6 square. It follows that

$$a_1 \geq 1, a_2 \geq 2, a_3 \geq 3, a_4 \geq 4, a_5 \geq 4, a_6 \geq 5, a_7 \geq 6, a_8 \geq 6, a_9 \geq 8.$$

So the total area of these nine subrectangles is at least

$$1 + 2 + 3 + 4 + 4 + 5 + 6 + 6 + 8 \geq 39,$$

which is larger than 36. This proves the given statement. ■

Exercise 4 (RMO 2006 P6,). Prove that there are infinitely many positive integers n such that $n(n+1)$ can be expressed as a sum of two positive squares in at least two different ways. (Here $a^2 + b^2$ and $b^2 + a^2$ are considered as the same representation.)

Walkthrough —

(a) Note that if two integers can be expressed as sum of two squares, then their product can also be expressed as a sum of two squares.

(b) Also note that if n is a perfect square, then the product $n(n+1)$ is a sum of two squares.

(c) If n is a perfect square, and it is a sum of two squares, that is, if $n = m^2$ and $n = a^2 + b^2$ hold, then

$$n(n+1) = (m^2)^2 + m^2 = (am - b)^2 + (a + bm)^2.$$

(d) Note that the perfect squares which are sum of two (nonzero) squares correspond to the Pythagorean triples, which are precisely the triples of

> the form, $(x^2-y^2, 2xy, x^2+y^2)$, or $(2xy, x^2-y^2, x^2+y^2)$, where x, y are integers satisfying $x > y \geq 1$.

Solution 5. Let k be a positive integer. Note that

$$(k^2-1)^2 + (2k)^2 = (k^2+1)^2$$

holds. Put $n = (k^2+1)^2$. Observe that n is a perfect square, and n is a sum of two squares. Note that the integer $n(n+1)$ can be expressed as a sum of two squares as

$$n(n+1) = ((k^2+1)^2)^2 + (k^2+1)^2. \tag{1}$$

Using the identity

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2,$$

it follows that the integer $n(n+1)$ can also be expressed as a sum of two squares as

$$n(n+1) = ((k^2-1)(k^2+1) - 2k)^2 + (k^2 - 1 + 2k(k^2+1))^2,$$

that is, as

$$n(n+1) = (k^4 - 2k - 1)^2 + (2k^3 + k^2 + 2k - 1)^2. \tag{2}$$

It suffices to find infinitely many positive integers k such that Eq. (1), Eq. (2) provide two different expressions of $n(n+1)$ as a sum of two positive squares.

Since the polynomials $k^4 - 2k - 1, 2k^3 + k^2 + 2k - 1$ have degrees higher than the degree of the polynomial k^2+1, it follows that for large enough values of k, the inequalities

$$k^4 - 2k - 1 > k^2 + 1, 2k^3 + k^2 + 2k - 1 > k^2 + 1$$

hold. Indeed,

$$\begin{aligned} k^4 - 2k - 1 - (k^2+1) &= \left(\frac{k^4}{3} - k^2\right) + \left(\frac{k^4}{3} - 2k\right) + \left(\frac{k^4}{3} - 2\right) \\ &> 0, \\ 2k^3 + k^2 + 2k - 1 - (k^2+1) &= 2k^3 - 1 + 2k - 1 \\ &> 0 \end{aligned}$$

hold if $k \geq 2$. Hence, for any integer $k \geq 2$, putting $n = (k^2+1)^2$, it follows that $n(n+1)$ can be expressed as the sum of two positive squares in at least two different ways. ■

Exercise 5 (RMO 2006 P7,). Let X be the set of all positive integers greater than or equal to 8, and let $f\colon X \to X$ be a function such that $f(x+y) = f(xy)$ for all $x \geq 4$, $y \geq 4$. If $f(8) = 9$, determine $f(9)$.

Walkthrough — Using the given condition, try to express $f(9)$ in terms of $f(8)$.

Solution 6. Note that

$$f(9) = f(4+5) = f(20) = f(4+16) = f(64),$$

and

$$f(8) = f(4+4) = f(16) = f(8+8) = f(64)$$

hold. Using the given condition $f(8) = 9$, it follows that $f(9) = 9$. ■