RMO 2005

Exercise 1 (RMO 2005 P1,). Let $ABCD$ be a convex quadrilateral. Let P, Q, R, S be the midpoints of AB, BC, CD, DA respectively such that triangles AQR and CSP are equilateral. Prove that $ABCD$ is a rhombus. Determine its angles.

Walkthrough —

(a) Show that PS, RQ are parallel and are of the same length, to conclude that $PQRS$ is a parallelogram.

(b) Using the fact that triangles AQR and CSP are equilateral, show that $\angle CSP, \angle AQR$ are equal.

(c) Use this to prove that AQ is parallel to CS, and are of the same length.

(d) Prove that AB is parallel to CD, and are of the same length, to conclude that $ABCD$ is a parallelogram.

(e) Since the medians AR, CS of $\triangle ACD$ are of the same length, show that $\angle RAC, \angle SCA$ are equal.

(f) Use this to prove that $\triangle RAC, \triangle SCA$ are congruent, and hence, $AD = CD$.

(g) Conclude that $ABCD$ is a rhombus.

(h) Show that the medians of $\triangle ACD$ are equal, to conclude that $\triangle ACD$ is equilateral.

(i) Determine the angles of the rhombus $ABCD$.

Solution 1. Since P, S are mid-points of AB, DA respectively, it follows that PS is parallel to BD and it is equal to $\frac{1}{2}BD$. Similarly, RQ is parallel to BD and it is equal to $\frac{1}{2}BD$. This shows that PS, RQ are equal and parallel to each other.

Thus $\angle CSP, \angle CUQ$ are equal. Since $\triangle CSP, \triangle AQR$ are equilateral, it follows that $\angle CSP, \angle AQR$ are equal to $60°$. This shows that $\angle CUQ$ is equal to $\angle AQR$. Consequently, AQ is parallel to CS, and they are equal. This implies that $AQCS$ is a parallelogram. Hence, AD is parallel to BC, and they

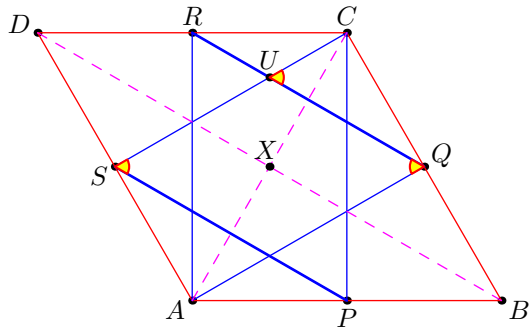


Figure 1: RMO 2005 P1, Exercise 1

are equal since S (resp. Q) is the mid-point of AD (resp. BC). Consequently, $ABCD$ is a parallelogram.

Since the medians AR, CS of $\triangle ACD$ are equal, it follows that $\angle RAC, \angle SCA$ are equal. This implies that $\triangle RAC, \triangle SCA$ are congruent, which gives $AS = RC$, and hence, $AD = CD$. Since $ABCD$ is a parallelogram and $AD = CD$, it follows that $ABCD$ is a rhombus.

Denote by X the point of intersection of the diagonals of $ABCD$. Note that DX is a median of $\triangle ACD$, and $DX = \frac{1}{2}BD = PS = CS$. Using the equality of the medians DX, CS of $\triangle ACD$, we get that $AC = AD$. Since AC, AD, CD are equal, it follows that $\triangle ACD$ is equilateral. Thus, $ABCD$ is a rhombus with $\angle ADC = 60°$. Moreover, we have

$$\angle ABC = 60°, \angle DAB = \angle DCB = 120°.$$

■

Exercise 2 (RMO 2005 P2,). If x, y are integers and 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$, then prove that 17 divides $xy - 12x + 15y$.

Walkthrough —

(a) Factorize $x^2 - 3xy + 2y^2 + x - y$ to show that

$$x \equiv y \pmod{17}, \quad \text{or } x \equiv 2y - 1 \pmod{17}$$

holds.

(b) Consider the above cases seperately, and use the divisibility of the other expression by 17 to obtain some congruence conditions on y. Using these conditions to read $xy - 12x + 15y$ modulo 17.

The content posted here and at this blog by Evan Chen are quite useful.

Solution 2. Let x, y be integers such that 17 divides both the expressions $x^2 - 2xy + y^2 - 5x + 7y$ and $x^2 - 3xy + 2y^2 + x - y$. Note that

$$x^2 - 3xy + 2y^2 + x - y = (x - y)(x - 2y + 1),$$

which is divisible by 17. It follows that

$$x \equiv y \bmod 17, \quad \text{or } x \equiv 2y - 1 \bmod 17$$

holds.

Let us consider the case that $x \equiv y \bmod 17$. It follows that

$$x^2 - 2xy + y^2 - 5x + 7y \equiv (x - y)^2 - 5x + 7y \equiv 2y \bmod 17.$$

Since 17 divides $x^2 - 2xy + y^2 - 5x + 7y$, we get $2y \equiv 0 \bmod 17$, which yields $x \equiv y \equiv 0 \bmod 17$, and hence 17 divides $xy - 12x + 15y$.

Let us consider the case that $x \equiv 2y - 1 \bmod 17$. Using $x^2 - 2xy + y^2 - 5x + 7y \equiv 0 \bmod 17$, we obtain

$$(2y - 1)^2 - 2(2y - 1)y + y^2 - 5(2y - 1) + 7y \equiv 0 \bmod 17,$$

which yields $y^2 - 5y + 6 \equiv 0 \bmod 7$. This implies that $(y-2)(y-3) \equiv 0 \bmod 17$. This shows that either $x \equiv 3 \bmod 17, y \equiv 2 \bmod 17$ holds, or $x \equiv 5 \bmod 17, y \equiv 3 \bmod 17$ holds. If $x \equiv 3 \bmod 17, y \equiv 2 \bmod 17$ holds, then

$$xy - 12x + 15y \equiv 6 - 36 + 30 \equiv 0 \bmod 17$$

holds. If $x \equiv 5 \bmod 17, y \equiv 3 \bmod 17$ holds, then we obtain

$$xy - 12x + 15y \equiv 15 - 60 + 45 \equiv 0 \bmod 17.$$

This proves that 17 divides $xy - 12x + 15y$. ■

Example 3. If a, b, c are three real numbers such that $|a - b| \geq |c|, |b - c| \geq |a|, |c - a| \geq |b|$, then prove that one of a, b, c is the sum of the other two.

Solution 3. The given inequalities are equivalent to $(a - b)^2 - c^2 \geq 0, (b - c)^2 - a^2 \geq 0, (c - a)^2 - b^2 \geq 0$, which yields

$$\begin{aligned}(a - b + c)(a - b - c) &\geq 0,\\ (b - c + a)(b - c - a) &\geq 0,\\ (c - a + b)(c - a - b) &\geq 0.\end{aligned}$$

Multiplying them, we obtain

$$-(b + c - a)^2(c + a - b)^2(a + b - c)^2 \geq 0,$$

which shows that $(b + c - a)(c + a - b)(a + b - c)$ is equal to 0. This proves the result. ■

Exercise 4 (RMO 2005 P4,). Find the number of all 5-digit numbers (in base 10) each of which contains the block 15 and is divisible by 15. (For example, 31545, 34155 are two such numbers.)

Solution 4. Note that any such number ends with 5 or 0, so is one of the following types: $abc15, ab150, ab155, a15b0, a15b5, 15ab0, 15ab5$. The 5-digit numbers of the form $abc15$ which are divisible by 5 are $10215, 10515, 10815, \ldots, 99915$, so there are $1 + \frac{1}{3}(999 - 102) = 300$ such numbers. The 5-digit numbers of the form $ab150$, divisible by 3 are $12150, 15150, 18150, \ldots, 99150$, so there are $1 + \frac{1}{3}(99 - 12) = 30$ such numbers. Similarly, there are $1 + \frac{1}{3}(97 - 10) = 30$ five digit numbers of the form $ab150$, divisible by 3. Also there are 30 numbers of the each of the following forms: $a15b0, a15b5$. For the form $15ab0$ (resp. $15ab5$), we get $1 + \frac{1}{3}(99 - 00) = 34$ (resp. $1 + \frac{1}{3}(97 - 01) = 33$) numbers. So there are exactly

$$300 + 4 \cdot 30 + 34 + 33 = 487$$

many 5-digit numbers which contain the block 15 and are divisible by 15. ■

Exercise 5 (RMO 2005 P6,). Determine all triples (a, b, c) of positive integers such that $a \leq b \leq c$ and

$$a + b + c + ab + bc + ca = abc + 1.$$

Solution 5. Let a, b, c be integers satisfying the given conditions. Note that the above equation can be rewritten as

$$(1 + a)(1 + b)(1 + c) = 2(abc + 1). \tag{1}$$

Note that if the inequalities

$$2^{\frac{1}{3}}a \geq a + 1, 2^{\frac{1}{3}}b \geq b + 1, 2^{\frac{1}{3}}c \geq c + 1$$

hold, then

$$(1 + a)(1 + b)(1 + c) \leq 2abc < 2(abc + 1)$$

holds, which is impossible. Also note that if $a \geq 4$, then

$$\frac{1}{2^{\frac{1}{3}} - 1} = 2^{\frac{2}{3}} + 2^{\frac{1}{3}} + 1 < \frac{8}{5} + \frac{7}{5} + 1 = 4$$

holds, which implies that

$$2^{\frac{1}{3}}a \geq a + 1, 2^{\frac{1}{3}}b \geq b + 1, 2^{\frac{1}{3}}c \geq c + 1.$$

This proves that $a \leq 3$. From Eq. (1), it follows that $a \neq 1$, and hence, a is equal to one of $2, 3$.

Also note that Eq. (1) does not hold if the inequalities

$$\sqrt{2a}\cdot b \geq \sqrt{1+a}(b+1), \sqrt{2a}\cdot c \geq \sqrt{1+a}(c+1)$$

hold. Observe that if

$$b \geq \begin{cases} 7 & \text{if } a=2, \\ 5 & \text{if } a=3, \end{cases}$$

holds, then we obtain

$$\frac{\sqrt{1+a}}{\sqrt{2a}-\sqrt{1+a}} = \begin{cases} \dfrac{\sqrt{3}}{2-\sqrt{3}} < 7 & \text{if } a=2, \\ \dfrac{2}{\sqrt{6}-2} < 5 & \text{if } a=3. \end{cases}$$

This shows that $b \leq 6$ if $a=2$, and $b \leq 4$ if $a=3$. Note that Eq. (1) is equivalent to

$$(2ab-(1+a)(1+b))c = (1+a)(1+b)-2,$$

which gives

$$\begin{aligned} c(b-3) &= 3b+1 \text{ if } a=2, \\ c(b-2) &= 2b+1 \text{ if } a=3. \end{aligned}$$

It follows that if $a=2$, then $b \neq 2$ and $b \neq 3$, and if $a=3$, then $b \neq 4$. This shows that (a,b,c) is equal to one of

$$(2,4,13), (2,5,8), (3,3,7).$$

Note that these triples satisfy Eq. (1). Consequently, the above triples are precisely all the solutions. ■

Remark. Note that Eq. (1) suggests to substitute

$$x = 1+a, y = 1+b, z = 1+c,$$

which yields

$$xyz = 2+2(x-1)(y-1)(z-1),$$

which can be rewritten as

$$xyz + 2(x+y+z) = 2(xy+yz+zx).$$

The above reduces to

$$2(x+y+z) + xy\left(\frac{z}{3}-2\right) + yz\left(\frac{x}{3}-2\right) + zx\left(\frac{y}{3}-2\right) = 0,$$

which shows that $x \leq 5$. Can one use the above to determine all the solutions?

Exercise 6 (RMO 2005 P7,). Let a, b, c be three positive real numbers such that $a+b+c=1$. Let

$$\lambda = \min\{a^3 + a^2bc, b^3 + ab^2c, c^3 + abc^2\}.$$

Prove that the roots of the equation $x^2 + x + 4\lambda = 0$ are real.

Solution 6. We need to show that the discriminant of $x^2+x+4\lambda$ is nonnegative, that is, $\lambda \leq \frac{1}{16}$ holds. On the contrary, let us assume that $\lambda > \frac{1}{16}$. This gives

$$a^3 + a^2bc > \frac{1}{16}, b^3 + ab^2c > \frac{1}{16}, c^3 + abc^2 > \frac{1}{16}.$$

Note that

$$a^3 + a^2bc = a^2(a+bc) = a^2(1-b-c+bc) = a^2(1-b)(1-c) > \frac{1}{16}$$

holds. Similarly, it follows that

$$b^2(1-c)(1-a) > \frac{1}{16}, c^2(1-a)(1-b) > \frac{1}{16}.$$

This implies that

$$(abc(1-a)(1-b)(1-c))^2 > \frac{1}{16^3},$$

which is impossible since

$$a(1-a) \leq \frac{1}{4}, b(1-b) \leq \frac{1}{4}, c(1-c) \leq \frac{1}{4}.$$

Consequently, we obtain $\lambda \leq \frac{1}{16}$, and hence the roots of the equation $x^2 + x + 4\lambda = 0$ are real. ■