RMO 2004

Exercise 1 (RMO 2004 P2,). Positive integers are written on all the faces of a cube, one on each. At each corner (vertex) of the cube, the product of the numbers on the faces that meet at the corner is written. The sum of the numbers written at all the corners is 2004. If T denotes the sum of the numbers on all the faces, find all the possible values of T.

Solution 1. Denote the integers on the top and bottom face of the cube by p, q respectively, and the integers written on four other faces by a, b, c, d. So the integers pab, pbc, pcd, pda are written at the four vertices of the face at the top, and the integers qab, qbc, qcd, qda are written at the four vertices of the face at the bottom. The sum of these eight integers is equal to 2004, that is,

$$(p+q)(ab+bc+cd+da) = 2004,$$

which gives

$$(p+q)(a+c)(b+d) = 2^2 \cdot 3 \cdot 167.$$

Note that $2, 3, 167$ are primes. Since a, b, c, d, p, q are positive integers, each of $p+q, a+c, b+d$ is at least 2, and exactly one of them has two prime factors (counting multiplicities). So the integers $p+q, a+c, b+d$ are equal to $2 \cdot 2, 3, 167$, or $2 \cdot 3, 2, 167$, or $2 \cdot 167, 2, 3$, or $3 \cdot 167, 2, 2$ in some order. Also note that there are positive integers a, b, c, d, p, q such that $p+q, a+c, b+d$ take these values. Thus the possible values of T are

$$\begin{aligned} 2 \cdot 2 + 3 + 167 &= 174, \\ 2 \cdot 3 + 2 + 167 &= 175, \\ 2 \cdot 167 + 2 + 3 &= 339, \\ 3 \cdot 167 + 2 + 3 &= 506. \end{aligned}$$

■

Exercise 2 (RMO 2004 P3,). Let α and β be the roots of the quadratic equation $x^2 + mx - 1 = 0$, where m is an odd integer. Let $\lambda_n = \alpha^n + \beta^n$, for $n \geq 0$. Prove that for $n \geq 0$,

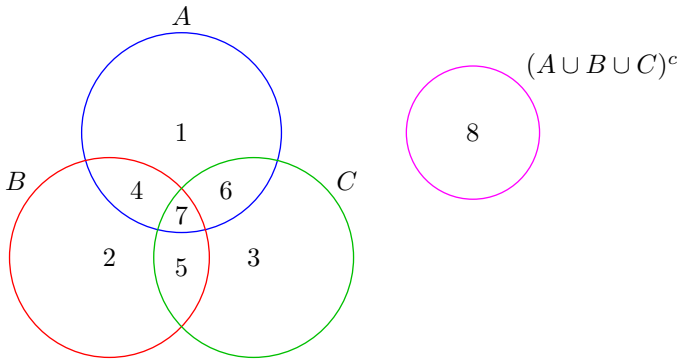


Figure 1: India RMO 2004, Exercise 3

1. λ_n is an integer, and
2. $\gcd(\lambda_n, \lambda_{n+1}) = 1$.

Solution 2. For any positive integer n, note that

$$(\alpha+\beta)(\alpha^n+\beta^n) = \alpha^{n+1}+\beta^{n+1}+\alpha\beta(\alpha^{n-1}+\beta^{n-1}),$$

which implies that

$$\begin{aligned}\alpha^{n+1}+\beta^{n+1} &= (\alpha+\beta)(\alpha^n+\beta^n) - \alpha\beta(\alpha^{n-1}+\beta^{n-1}) \\ &= -m(\alpha^n+\beta^n) + (\alpha^{n-1}+\beta^{n-1}),\end{aligned}$$

which yields that

$$\lambda_{n+1} = -m\lambda_n + \lambda_{n-1}. \tag{1}$$

For any integer $n \geq 1$, let $P(n)$ denote the statement that $\lambda_0, \lambda_1, \ldots, \lambda_n$ are integers, and $\gcd(\lambda_{n-1}, \lambda_n) = 1$ holds. Note that $\lambda_0 = 2, \lambda_1 = -m$. Since m is an odd integer, it follows that $P(1)$ holds. Assume that the statement $P(k)$ holds for some positive integer k. Using Eq. (1), and using the induction hypothesis, it follows that λ_{k+1} is an integer. Using Eq. (1) once again, note that any common divisor of λ_k and λ_{k+1} divides λ_{k-1}, and hence is also a common divisor of λ_{k-1}, λ_k. Applying the induction, we conclude that the integers λ_k and λ_{k+1} are relatively prime. This proves that $P(k+1)$ holds. By the principle of induction, the statement $P(n)$ holds for any positive integer n. This completes the proof. ■

Exercise 3 (RMO 2004 P4,). Prove that the number of triples (A, B, C) where A, B, C are subsets of $\{1, 2, \ldots, n\}$ such that $A \cap B \cap C = \emptyset$, $A \cap B \neq \emptyset$, $B \cap C \neq \emptyset$ is $7^n - 2 \cdot 6^n + 5^n$.

Walkthrough — Establish a convenient bijection between the triples of subsets of $\{1, 2, \ldots, n\}$, and the maps from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, 8\}$. (Hint: Use Venn diagram.) Use this bijection to count the number of triples satisfying the given conditions.

Solution 3. Note that the triples (A, B, C) of subsets of $\{1, 2, \ldots, n\}$ are in one-to-one correspondense with the maps from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, 8\}$. One such correspondense is given by sending (A, B, C) to the map $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, 8\}$, which takes the values $1, 2, \ldots, 8$ at the following subsets

$$A \setminus (B \cup C), B \setminus (C \cup A), C \setminus (A \cup B),$$
$$(A \cap B) \setminus (A \cap B \cap C), (B \cap C) \setminus (A \cap B \cap C), (C \cap A) \setminus (A \cap B \cap C),$$
$$\{1, 2, \ldots, n\} \setminus (A \cup B \cup C),$$

of $\{1, 2, \ldots, n\}$ respectively. Hence, the number of triples (A, B, C) satisfying the given conditions is equal to the number of maps $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, 8\}$, satisfying

$$f^{-1}(7) = \emptyset, f^{-1}(4) \neq \emptyset, f^{-1}(5) \neq \emptyset.$$

Let $\mathcal{F}$ denote the set of maps $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, 8\}$. Applying the inclusion-exclusion principle, we obtain

$$\begin{aligned}
&|\{f \in \mathcal{F} \mid f^{-1}(7) = \emptyset, f^{-1}(4) \neq \emptyset, f^{-1}(5) \neq \emptyset\}| \\
&= \left|\left(\{f \in \mathcal{F} \mid f^{-1}(7) = \emptyset, f^{-1}(4) = \emptyset\} \cup \{f \in \mathcal{F} \mid f^{-1}(7) = \emptyset, f^{-1}(5) = \emptyset\}\right)^c\right| \\
&= |\{f \in \mathcal{F} \mid f^{-1}(7) = \emptyset\}| \\
&\quad - |\{f \in \mathcal{F} \mid f^{-1}(7) = \emptyset, f^{-1}(4) = \emptyset\}| \\
&\quad - |\{f \in \mathcal{F} \mid f^{-1}(7) = \emptyset, f^{-1}(5) = \emptyset\}| \\
&\quad + |\{f \in \mathcal{F} \mid f^{-1}(7) = \emptyset, f^{-1}(4) = \emptyset, f^{-1}(5) = \emptyset\}| \\
&= 7^n - 2 \cdot 6^n + 5^n,
\end{aligned}$$

where in the above, for a subset $\mathcal{E}$ of $\mathcal{F}$, the complement of $\mathcal{E}$ in $\mathcal{F}$ is denoted by $\mathcal{E}^c$. This completes the proof. ■

Exercise 4 (RMO 2004 P6, , Australian MO 1982)**.** Let $(p_1, p_2, p_3, \ldots, p_n, \ldots)$ be a sequence of primes, defined by $p_1 = 2$ and for $n \geq 1$, p_{n+1} is the largest prime factor of $p_1 p_2 \cdots p_n + 1$. Prove that $p_n \neq 5$ for any n.

Walkthrough —

(a) Show that $p_1 p_2 p_3 \cdots p_n + 1$ is odd for any $n \geq 1$, and p_n is odd for any $n \geq 2$. Deduce that $p_1 p_2 p_3 \ldots p_n + 1$ is not a multiple of 3. (If you are stuck, then does verifying this statement for small values of n help?)

(b) What can be said about the smallest prime divisor of $p_1 p_2 p_3 \ldots p_n + 1$?

(c) If it is a power of 5, then $p_1p_2p_3\dots p_n$ is divisible by 4. Arrive at a contradiction.

Solution 4. Note that $p_1p_2\dots p_n+1$ is odd for any $n\geq 1$, and hence p_n is odd for any $n\geq 2$. Since $p_1=2$ and $p_2=3$, it follows that for any $n\geq 2$, the integer $p_1p_2\dots p_n+1$ is not divisible by any one of 2 and 3. So the least prime divisor of $p_1p_2\dots p_n+1$ is at least 5 for any $n\geq 2$. If possible, suppose 5 is the largest prime divisor of $p_1p_2\dots p_n+1$ for some integer $n\geq 2$. This yields

$$p_1p_2\dots p_n+1=5^k$$

for some $k\geq 1$. This implies that 4 divides $p_1p_2\cdots p_n$, which is impossible since $p_1=2$, and p_r is odd for any integer $r\geq 2$. This shows that p_{n+1} is not equal to 5 for any integer $n\geq 2$. Consequently, it follows that $p_n\neq 5$ for any integer $n\geq 1$. ■

Exercise 5 (RMO 2004 P7,). Let x and y be positive real numbers such that $y^3+y\leq x-x^3$. Prove that

1. $y<x<1$ and
2. $x^2+y^2<1$.

Solution 5. Note that $x-y\geq x^3+y^3>0$ holds, which shows that $x>y$. Also note that $x-x^3\geq y^3+y>0$ holds, which implies that $x(1-x^2)>0$. Since x is positive, we obtain $x<1$. This gives $0<y<x<1$.

Write $x=y+t$ with $t>0$. The inequality $y^3+y\leq x-x^3$ yields

$$2y^3+3y^2t+3yt+t^3\leq t.$$

Note that the inequality $x^2+y^2<1$ is equivalent to $2y^2+2yt+t^2<1$. Since $y+t$ is positive, it is equivalent to $(y+t)(2y^2+2yt+t^2)<y+t$. Observe that

$$\begin{aligned}(y+t)(2y^2+2yt+t^2)&=2y^3+4y^2t+3yt^2+t^3\\&\leq y^2t+3yt^2+t-3yt\\&=y^2t-3yt(1-t)+t\\&<y+t\quad(\text{using } 0<y,t<1).\end{aligned}$$

This completes the proof. ■